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| Name: |
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- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

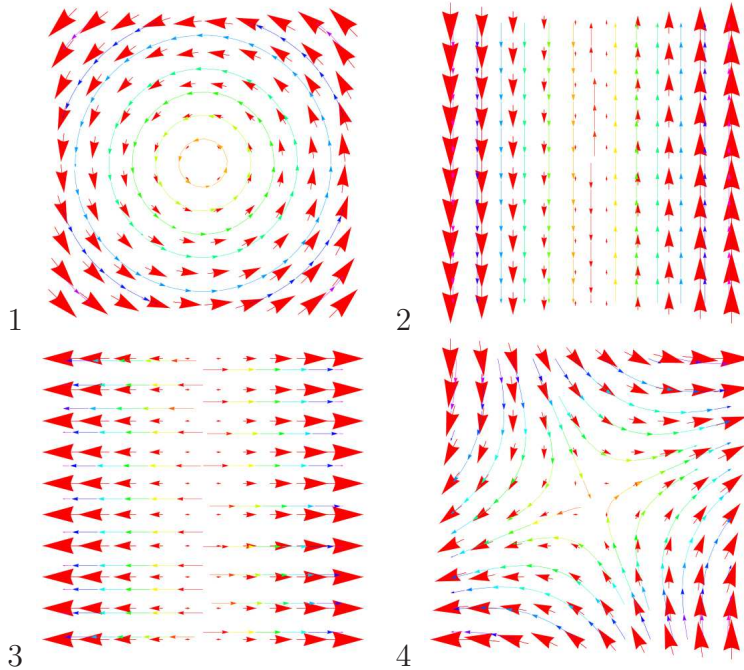
|        |  |     |
|--------|--|-----|
| 1      |  | 20  |
| 2      |  | 10  |
| 3      |  | 10  |
| 4      |  | 10  |
| 5      |  | 10  |
| 6      |  | 10  |
| 7      |  | 10  |
| 8      |  | 10  |
| 9      |  | 10  |
| 10     |  | 10  |
| 11     |  | 10  |
| 12     |  | 10  |
| 13     |  | 10  |
| Total: |  | 140 |

Problem 1) (20 points)

- 1)  T  F The line  $\vec{r}(t) = \langle t, t, -t \rangle$  is contained in the plane  $x + y + z = 1$ .
- 2)  T  F The quadratic surface  $x^2 + y^2 = z^2$  is an elliptic paraboloid.
- 3)  T  F If  $\vec{T}(t), \vec{B}(t), \vec{N}(t)$  are the unit tangent, normal and binormal vectors of a curve with  $\vec{r}'(t) \neq 0$  everywhere, then  $\vec{T}(t) \cdot \vec{B}(t) \times \vec{N}(t)$  is always equal to 1 or  $-1$ .
- 4)  T  F If  $|\vec{u} \times \vec{v}| = 0$ , then  $\text{Proj}_{\vec{u}}(\vec{v}) = \vec{v}$ .
- 5)  T  F There is a vector field  $\vec{F}(x, y)$  which has the property  $\text{curl}(\vec{F}) = \text{div}(\vec{F}) = 1$ , where  $\text{curl}(\vec{F})(x, y) = Q_x(x, y) - P_y(x, y)$  and  $\text{div}(\vec{F})(x, y) = P_x(x, y) + Q_y(x, y)$ .
- 6)  T  F The acceleration vector  $\vec{r}''(t) = \langle x(t), y(t) \rangle$  is always in the plane spanned by the vector  $\vec{r}(t)$  and the velocity vector  $\vec{r}'(t)$ .
- 7)  T  F For every curve on the unit sphere, the curvature is constant and equal to 1.
- 8)  T  F If a smooth function  $f(x, y)$  has no maximum nor minimum, then it does not have a critical point.
- 9)  T  F The linearization  $L(x, y)$  of a cubic function  $f(x, y) = x^3 + y^3$  is the function  $L(x, y) = 3x^2 + 3y^2$ .
- 10)  T  F If  $\vec{F}(x, y)$  is a gradient field  $\vec{F} = \nabla f$  and  $\vec{r}(t)$  is a flow line satisfying  $\vec{r}'(t) = \vec{F}(\vec{r}(t))$  then  $\frac{d}{dt} f(\vec{r}(t)) = |\vec{F}|^2(\vec{r}(t))$ .
- 11)  T  F If  $f + g$  and  $f - g$  have a common critical point  $(a, b)$ , then this point is a critical point of both  $f$  and  $g$ .
- 12)  T  F Assume a vector field  $\vec{F}(x, y, z)$  is a gradient field, then  $\int_C \vec{F} \cdot d\vec{r} = 0$  where  $C$  is the intersection of  $x^2 + y^2 = 1$  with  $z = 1$ .
- 13)  T  F If the flux of vector field is zero through any surface  $S$  in space, then the divergence of the field is zero everywhere in space.
- 14)  T  F The curl of a gradient field  $\vec{F}(x, y, z) = \nabla f(x, y, z)$  is zero, if  $f(x, y, z) = \sqrt{x^{10} + y^{10} z^2}$ .
- 15)  T  F The line integral of the curl of a vector field  $\vec{F}(x, y, z) = \langle x, y, z \rangle$  along a circle in the  $xy$ - plane is zero.
- 16)  T  F For a solid  $E$  which is rotationally symmetric around the  $z$ -axes, the integral  $\iiint_E \sqrt{x^2 + y^2} dx dy dz$  is equal to the volume of the solid.
- 17)  T  F The curvature of the curve  $\vec{r}(t) = \langle 1 + 2 \cos(1+t), 1 + 2 \sin(1+t) \rangle$  is constant equal to  $1/2$  everywhere.
- 18)  T  F The directional derivative of  $f(x, y, z) = \text{div}(\vec{F}(x, y, z))$  of the divergence of the vector field  $\vec{F} = \langle P, Q, R \rangle$  in the direction  $\vec{v} = \langle 1, 0, 0 \rangle$  is  $P_{xx} + Q_{xy} + R_{xz}$ .
- 19)  T  F  $\int_0^{2\pi} \int_0^{2\pi} r d\theta dr = \int_0^{2\pi} \int_0^{2\pi} 1 dx dy$ .
- 20)  T  F The set  $\{\phi = \pi, \rho > 0\}$  in spherical coordinates is the negative  $z$ -axis.

Problem 2) (10 points) No justifications are necessary.

a) (4 points) Match the vector fields with the definitions



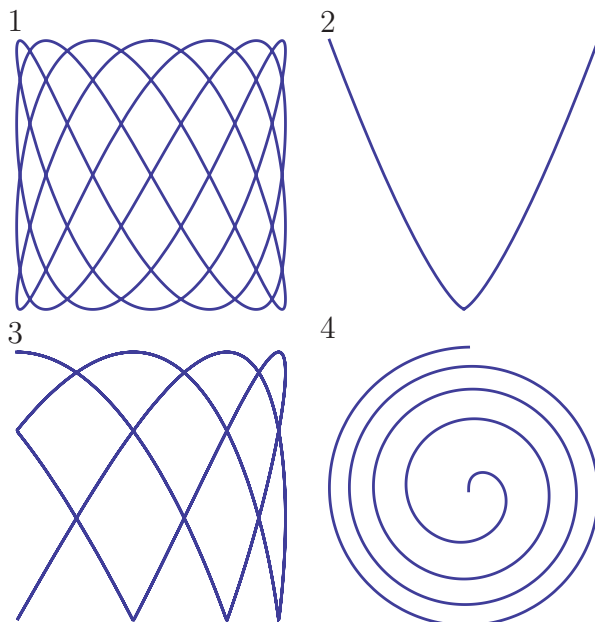
| Enter 1-4 | vector field                                   |
|-----------|--|
|           | $\vec{F}(x, y) = \langle x + y, x - y \rangle$ |
|           | $\vec{F}(x, y) = \langle 0, x \rangle$         |
|           | $\vec{F}(x, y) = \langle -y, x \rangle$        |
|           | $\vec{F}(x, y) = \langle x, 0 \rangle$         |

b) (3 points) Match the partial differential equations (PDE's) with their names

- 1) Wave equation
- 2) Heat equation
- 3) Transport equation
- 4) Burgers equation

| Enter 1-4 | PDE                   |
|-----------|-----------------------|
|           | $u_t - u_{xx} = 0$    |
|           | $u_{tt} - u_{xx} = 0$ |
|           | $u_t - u_x = 0$       |

c) (3 points) Match the curves



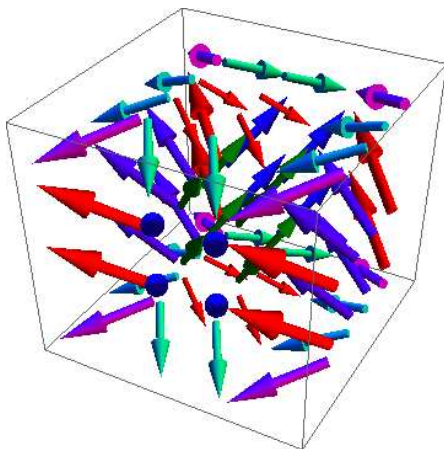
| Enter 1-4 | Parametrized curve  |
|-----------|---|
|           | $\vec{r}(t) = \langle \cos(4t), \sin(7t) \rangle$                 |
|           | $\vec{r}(t) = \langle \sqrt{t} \sin(t), \sqrt{t} \cos(t) \rangle$ |
|           | $\vec{r}(t) = \langle  \cos(4t) ,  \sin(7t)  \rangle$             |
|           | $\vec{r}(t) = \langle t^3, t^4 \rangle$                           |

Problem 3) (10 points) No justifications are necessary

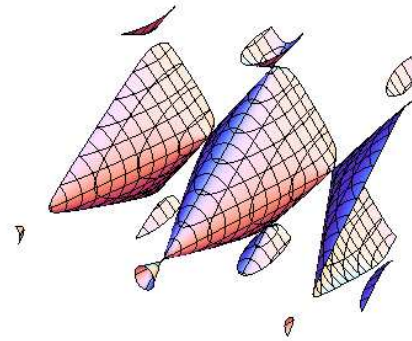
a) (6 points) Check the boxes which apply. Leave the other boxes empty. The expression "involves XYZ" means that the formulation of the statement contains the object XYZ somewhere.

| Statement                    | involves a curve | involves a surface | involves a vector field |
|------------------------------|------------------|--------------------|-------------------------|
| Stokes theorem               |                  |                    |                         |
| Divergence theorem           |                  |                    |                         |
| Lagrange equations           |                  |                    |                         |
| Fund. theorem line integrals |                  |                    |                         |
| Surface area formula         |                  |                    |                         |
| Curvature formula            |                  |                    |                         |

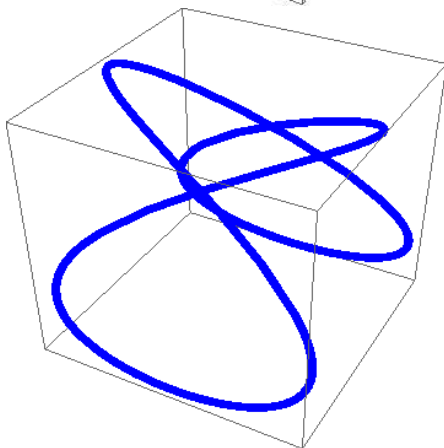
b) (4 points) Match the objects with their definitions



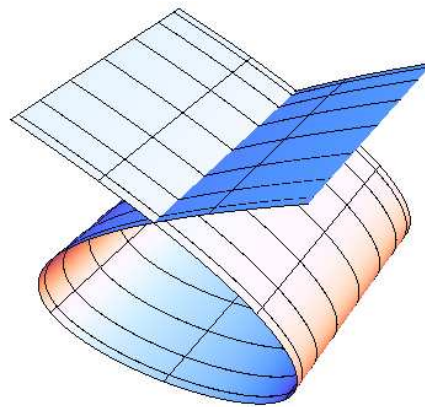
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2



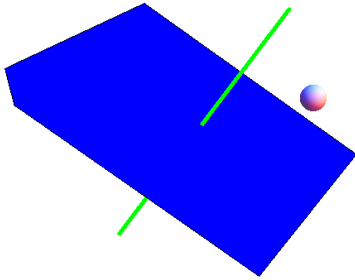
3



4

| Enter 1-4 | object definition  |
|-----------|--|
|           | $\vec{r}(t) = \langle \cos(3t), \sin(t), \cos(2t) \rangle$       |
|           | $\cos(3x) + \sin(y) + \cos(2z) = 1$                              |
|           | $\vec{r}(t, s) = \langle \cos(3t), \sin(s), \cos(2t) \rangle$    |
|           | $\vec{F}(x, y, z) = \langle \cos(3x), \sin(y), \cos(2z) \rangle$ |

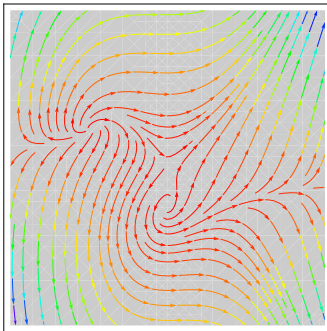
Problem 4) (10 points)



a) (5 Points) Write down a parametrization  $\vec{r}(t)$  of the line which is perpendicular to the plane  $x + 2y + z = 0$  and which passes through the origin.

b) (5 points) Find the distance of this line to the point  $(3, 4, 5)$ .

Problem 5) (10 points)



Find the place where the curl

$$f(x, y) = \text{curl}(\vec{F})(x, y) = Q_x(x, y) - P_y(x, y)$$

of the vector field

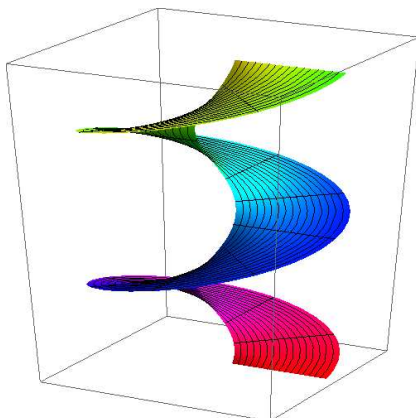
$$\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \langle x + y^2 + y, x^2y + 2x + y^2 \rangle$$

is maximal under the constraint that the divergence

$$g(x, y) = \text{div}(\vec{F})(x, y) = P_x(x, y) + Q_y(x, y) .$$

is equal to 1. Find the functions  $f, g$  and solve the problem using Lagrange.

Problem 6) (10 points)



a) (5 points) Find the surface area of the surface

$$\vec{r}(s, t) = \langle s \cos(t), s \sin(t), t \rangle$$

with  $1 \leq s \leq 2, 0 \leq t \leq 4\pi$ .

b) (5 points) Find the arc length of the curve

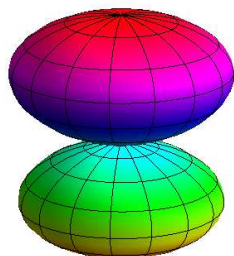
$$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$$

with  $0 \leq t \leq 4\pi$ .

**Hint.** You can use without derivation the during lecture derived integral  $\int \sqrt{x^2 + 1} dx =$

$(x\sqrt{x^2 + 1} + \operatorname{arcsinh}(x))/2$  and you can leave terms like  $\operatorname{arcsinh}(2)$ .

Problem 7) (10 points)

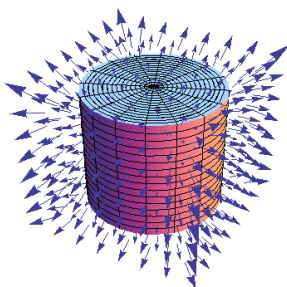


Find the volume of the solid given in spherical coordinates as

$$\rho(\phi, \theta) \leq \cos^2(\phi) .$$

with  $0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi$ .

Problem 8) (10 points)



Find the flux  $\iint_S \vec{F} \cdot d\vec{S}$  of the vector field

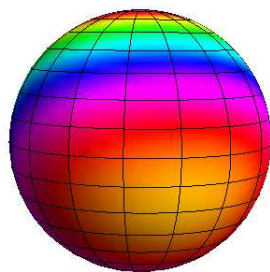
$$\vec{F}(x, y, z) = \langle x^3, y^3, z + (1 - x^2 - y^2)(1 - z^2) \rangle$$

through the boundary  $S$  of the solid cylinder

$$E : x^2 + y^2 \leq 1, z^2 \leq 1 .$$

The boundary of  $S$  of the solid  $E$  is oriented outwards as usual.

Problem 9) (10 points)



Where on the sphere is the function

$$f(\phi, \theta) = \sin(\phi) + \sin(\theta)$$

extremal? Find all maxima, minima and saddle points of  $f$  as well as the global maxima and minima.

**Remark.** The variables  $\phi, \theta$  are the usual spherical coordinates variables. You are welcome of course to write  $f(x, y) = \sin(x) + \sin(y)$  and look for solutions  $0 \leq x \leq \pi, 0 \leq y < 2\pi$ .

Problem 10) (10 points)

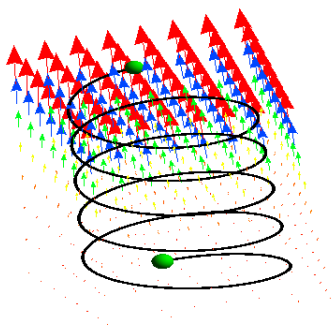


The "sin-log" function  $\sin(x)/\log(x)$  has no known antiderivative. Determined to overcome this obstacle, we nevertheless integrate

$$\int_0^1 \int_{e^y}^e \frac{\sin(x)}{\log(x)} dx dy .$$

Note that  $\log(x)$  denotes the natural logarithm. The  $\ln$  notation is for greenhorns.

Problem 11) (10 points)

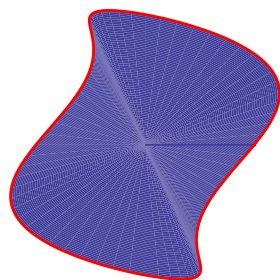


Find the line integral

$$\int_0^{10\pi} \vec{F} \cdot d\vec{r} ,$$

where  $\vec{F}(x, y, z) = \langle x^2 + 1, y^2, z^3 + x^2 \rangle$  and where  $\vec{r}(t)$  is the spiral  $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ .

Problem 12) (10 points)

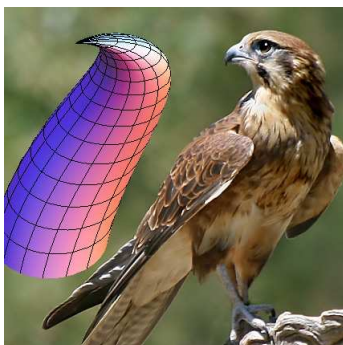


Find the area of the region enclosed by the curve

$$\vec{r}(t) = \langle \cos(t) + \frac{\sin(3t)}{3}, 3 \sin(t) \rangle$$

where  $0 \leq t \leq 2\pi$ .

Problem 13) (10 points)



Find the flux  $\iint_S \text{curl}(F) \, dS$  of the curl of the vector field

$$\vec{F}(x, y, z) = \langle -2y, 2x, 6e^{xy^2} \rangle$$

through the **falcon** surface

$$\vec{r}(s, t) = \left\langle \cos(t) \sin(s) + \frac{\sin(3s)}{2}, \sin(t) \sin(s) + \sin(2s), 4 \cos(s) \right\rangle$$

parametrized by  $0 \leq t < 2\pi$  and  $0 \leq s \leq \pi/2$ .