Lecture 21: Greens theorem

Green's theorem is the second and last integral theorem in two dimensions. This entire section deals with multivariable calculus in 2D, where we have 2 integral theorems, the fundamental theorem of line integrals and Greens theorem. First two reminders:

If $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ is a vector field and $C : \vec{r}(t) = \langle x(t), y(t) \rangle, t \in [a,b]$ is a curve, the **line integral**

$$\int_C \vec{F}\, d\vec{r} = \int_a^b \vec{F}(x(t),y(t)) \cdot \vec{r}\,'(t)\, dt$$

measures the **work** done by the field $\vec{F}$ along the path C.

The **curl** of a two dimensional vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ is the scalar field

$$\operatorname{curl}(F)(x,y) = Q_x(x,y) - P_y(x,y) \; .$$

The $\operatorname{curl}(F)$ measures the **vorticity** of the vector field.

One can write $\nabla \times \vec{F} = \operatorname{curl}(\vec{F})$ because the two dimensional cross product of (∂_x, ∂_y) with $\vec{F} = \langle P, Q \rangle$ is the scalar $Q_x - P_y$.

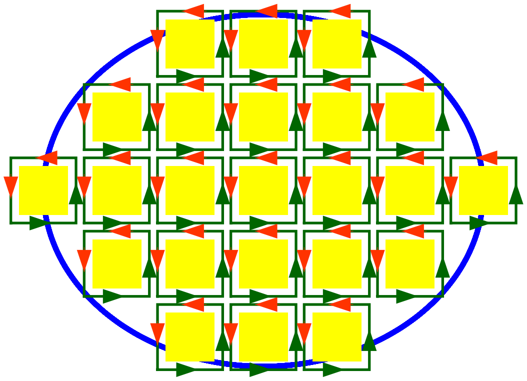
1 For $\vec{F}(x,y) = \langle -y, x \rangle$, we have $\operatorname{curl}(F)(x,y) = 2$. For $\vec{F}(x,y) = \langle x^3 + y^2, y^3 + x^2 y \rangle$, we have $\operatorname{curl}(F)(x,y) = 2xy - 2y$.

2 If $\vec{F}(x,y) = \nabla f$ is a gradient field then the curl is zero because if $P(x,y) = f_x(x,y), Q(x,y) = f_y(x,y)$ and $\operatorname{curl}(F) = Q_x - P_y = f_{yx} - f_{xy} = 0$ by Clairaut's theorem. The field $\vec{F}(x,y) = \langle x+y, yx \rangle$ for example is not a gradient field because $\operatorname{curl}(F) = y - 1$ is not zero.

Green's theorem: If $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ is a smooth vector field and R is a region for which the boundary C is a curve parametrized so that R is "to the left", then

$$\int_C \vec{F} \cdot d\vec{r} = \int \int_G \operatorname{curl}(F)\; dxdy \; .$$

Proof. Consider a square $G = [x, x+h] \times [y, y+h]$ with small $h > 0$. The line integral of $\vec{F} = \langle P, Q \rangle$ along the boundary is $\int_0^h P(x+t,y)dt + \int_0^h Q(x+h,y+t)\, dt - \int_0^h P(x+t,y+h)\, dt - \int_0^h Q(x,y+t)\, dt$. It measures the "circulation" at the place (x,y). Because $Q(x+h,y) - Q(x,y) \sim Q_x(x,y)h$ and $P(x,y+h) - P(x,y) \sim P_y(x,y)h$, the line integral is $(Q_x - P_y)h^2 \sim \int_0^h \int_0^h \operatorname{curl}(F)\; dxdy$. Now take a region G with area $|G|$ and chop it into small squares of size h. We need about $|G|/h^2$ such squares. Summing up all the line integrals around the boundaries is the sum of the line integral along the boundary of G because of the cancellations in the interior. On the boundary, it is a Riemann sum of the line integral along the boundary. The sum of the curls of the squares is a Riemann sum approximation of the double integral $\int \int_G \operatorname{curl}(F)\; dxdy$. In the limit $h \to 0$, we obtain Greens theorem.



George Green lived from 1793 to 1841. He was a physicist, a self-taught mathematician as well as a miller. His work greatly contributed to modern physics.

3 If $\vec{F}$ is a gradient field then both sides of Green's theorem are zero: $\int_C \vec{F} \cdot \vec{dr}$ is zero by the fundamental theorem for line integrals. and $\int\int_G \text{curl}(F) \cdot dA$ is zero because $\text{curl}(F) = \text{curl}(\text{grad}(f)) = 0$.

The already established Clairaut identity

$$\text{curl}(\text{grad}(f)) = 0$$

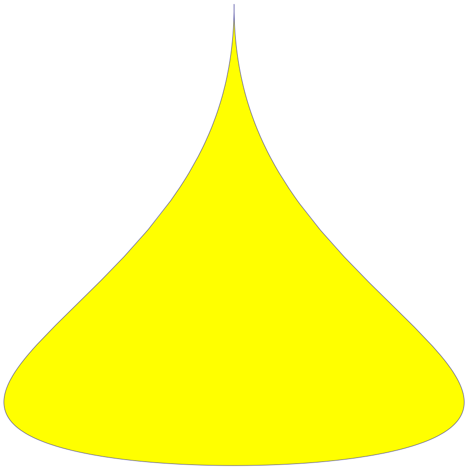
can also be remembered by writing $\text{curl}(\vec{F}) = \nabla \times \vec{F}$ and $\text{curl}(\nabla f) = \nabla \times \nabla f$. Use now that cross product of two identical vectors is 0. Working with ∇ as a vector is called **nabla calculus**.

4 Find the line integral of $\vec{F}(x,y) = \langle x^2 - y^2, 2xy \rangle = \langle P, Q \rangle$ along the boundary of the rectangle $[0,2] \times [0,1]$. Solution: $\text{curl}(\vec{F}) = Q_x - P_y = 2y - 2y = -4y$ so that $\int_C \vec{F}\, \vec{dr} = \int_0^2 \int_0^1 4y\, dydx = 2y^2|_0^1 x|_0^2 = 4$.

5 Find the area of the region enclosed by

$$\vec{r}(t) = \langle \frac{\sin(\pi t)^2}{t}, t^2 - 1 \rangle$$

for $-1 \leq t \leq 1$. To do so, use Greens theorem with the vector field $\vec{F} = \langle 0, x \rangle$.



6 Green's theorem allows to express the coordinates of the **centroid** = center of mass

$$(\int\int_G x\, dA/A, \int\int_G y\, dA/A)$$

using line integrals. With the vector field $\vec{F} = \langle 0, x^2 \rangle$ we have

$$\int\int_G x\, dA = \int_C \vec{F}\ \vec{dr}\ .$$

7 An important application of Green is the **computation of area**. Take a vector field like $\vec{F}(x,y) = \langle P, Q \rangle = \langle -y, 0 \rangle$ or $\vec{F}(x,y) = \langle 0, x \rangle$ which has vorticity $\text{curl}(\vec{F})(x,y) = 1$. For $\vec{F}(x,y) = \langle 0, x \rangle$, the right hand side in Green's theorem is the **area** of G:

$$\text{Area}(G) = \int_C x(t)\dot{y}(t)\, dt\ .$$

8 Let G be the region under the graph of a function $f(x)$ on $[a,b]$. The line integral around the boundary of G is 0 from $(a,0)$ to $(b,0)$ because $\vec{F}(x,y) = \langle 0, 0 \rangle$ there. The line integral is also zero from $(b,0)$ to $(b, f(b))$ and $(a, f(a))$ to $(a,0)$ because $N = 0$. The line integral along the curve $(t, f(t))$ is $-\int_a^b \langle -y(t), 0 \rangle \cdot \langle 1, f'(t) \rangle\, dt = \int_a^b f(t)\, dt$. Green's theorem confirms that this is the area of the region below the graph.

It had been a consequence of the fundamental theorem of line integrals that

If $\vec{F}$ is a gradient field then $\text{curl}(F) = 0$ everywhere.

Is the converse true? Here is the answer:

A region R is called **simply connected** if every closed loop in R can continuously be pulled together within R to a point inside R.

If $\text{curl}(\vec{F}) = 0$ in a simply connected region G, then $\vec{F}$ is a gradient field.

Proof. Given a closed curve C in G enclosing a region R. Green's theorem assures that $\int_C \vec{F}\ \vec{dr} = 0$. So $\vec{F}$ has the closed loop property in G. This is equivalent to the fact that line integrals are path independent. In that case $\vec{F}$ is therefore a gradient field: one can get $f(x,y)$ by taking the line integral from an arbitrary point O to (x,y). In the homework, you look at an example of a not simply connected region where the $\text{curl}(\vec{F}) = 0$ does not imply that $\vec{F}$ is a gradient field.

An engineering application of Greens theorem is the **planimeter**, a mechanical device for measuring areas. We will demonstrate it in class. Historically it had been used in medicine to measure the size of the cross-sections of tumors, in biology to measure the area of leaves or wing sizes of insects, in agriculture to measure the area of forests, in engineering to measure the size of profiles. There is a vector field $\vec{F}$ associated to a planimeter which is obtained by placing a unit vector perpendicular to the arm).
One can prove that $\vec{F}$ has vorticity 1. The planimeter calculates the line integral of $\vec{F}$ along a given curve. Green's theorem assures it is the area.

Homework

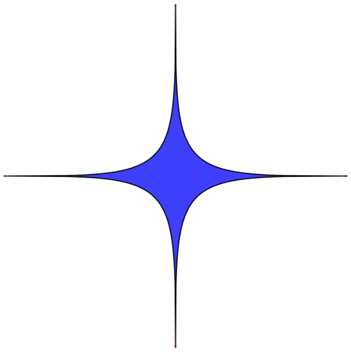
1 Calculate the line integral $\int_C \vec{F}\ d\vec{r}$ with $\vec{F} = \langle 2y + x\sin(y), x^2\cos(y) - 3y^{200\sin(y)}\rangle$ along a triangle C with edges $(0,0)$, $(\pi, 0)$ and (π, π).

2 Evaluate the line integral of the vector field $\vec{F}(x,y) = \langle 2xy^2, 2x^2\rangle$ along the rectangle with vertices $(0,0), (2,0), (2,3), (0,3)$.

3 Find the area of the region bounded by the **hypocycloid**

$$\vec{r}(t) = \langle 2\cos^3(t), 2\sin^3(t)\rangle$$

using Green's theorem. The curve is parameterized by $t \in [0, 2\pi]$.



4 Let G be the region $x^6 + y^6 \leq 1$. Compute the line integral of the vector field $\vec{F}(x,y) = \langle x^{800} + \sin(x), y^{12}\rangle$ along the boundary.

5 This is a classic: let $\vec{F}(x,y) = \langle -y/(x^2+y^2), x/(x^2+y^2)\rangle$. Let $C : \vec{r}(t) = \langle \cos(t), \sin(t)\rangle, t \in [0, 2\pi]$.
a) Compute $\int_C \vec{F} \cdot d\vec{r}$.
b) Show that $\text{curl}(\vec{F}) = 0$ everywhere for $(x,y) \neq (0,0)$.
c) Let $f(x,y) = \arctan(y/x)$. Verify that $\nabla f = \vec{F}$.
d) Why do a) and b) not contradict the fact that a gradient field has the closed loop property? Why does *a*) and *b*) not contradict Green's theorem?

Lecture 22: Curl and Divergence

We have seen the curl in two dimensions. It was the scalar field $\text{curl}(F) = Q_x - P_y$. By Greens theorem, it had been the average work of the field done along a small circle of radius r around the point in the limit when the radius of the circle goes to zero. Greens theorem has explained what the curl is. In three dimensions, the curl is a vector:

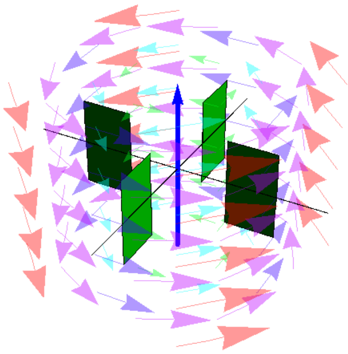
The **curl** of a vector field $\vec{F} = \langle P, Q, R \rangle$ is defined as the vector field

$$\text{curl}(P, Q, R) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \ .$$

Invoking nabla calculus, we can write $\text{curl}(\vec{F}) = \nabla \times \vec{F}$. Note that the third component of the curl is for fixed z just the two dimensional vector field $\vec{F} = \langle P, Q \rangle$ is $Q_x - P_y$. While the curl in 2 dimensions is a scalar field, it is a vector in 3 dimensions. In n dimensions, it would have $n(n-1)/2$ components, the number of 2-dimensional coordinate planes in n dimensions. The curl measures the "vorticity" of the field.

If a field has zero curl everywhere, the field is called **irrotational**.

The curl is often visualized using a "paddle wheel". If you place such a wheel into the field into the direction v, its rotation speed of the wheel measures the quantity $\vec{F} \cdot \vec{v}$. Consequently, the direction in which the wheel turns fastest, is the direction of $\text{curl}(\vec{F})$. Its angular velocity is the length of the curl. The wheel could actually be used to measure the curl of the vector field at any point. In situations with large vorticity like in a tornado, one can "see" the direction of the curl near the vortex center.



In two dimensions, we had two derivatives, the gradient and curl. In three dimensions, there are three fundamental derivatives, the **gradient**, the **curl** and the **divergence**.

The **divergence** of $\vec{F} = \langle P, Q, R \rangle$ is the scalar field $\text{div}(\langle P, Q, R \rangle) = \nabla \cdot \vec{F} = P_x + Q_y + R_z$.

The divergence can also be defined in two dimensions, but it is not fundamental. We want in n dimensions to have n fundamental derivatives and for each a fundamental theorem.

The **divergence** of $\vec{F} = \langle P, Q \rangle$ is $\mathrm{div}(P,Q) = \nabla \cdot \vec{F} = P_x + Q_y$.

In two dimensions, the divergence is just the curl of a -90 degrees rotated field $\vec{G} = \langle Q, -P \rangle$ because $\mathrm{div}(\vec{G}) = Q_x - P_y = \mathrm{curl}(\vec{F})$. The divergence measures the "expansion" of a field. If a field has zero divergence everywhere, the field is called **incompressible**.

With the "vector" $\nabla = \langle \partial_x, \partial_y, \partial_z \rangle$, we can write $\mathrm{curl}(\vec{F}) = \nabla \times \vec{F}$ and $\mathrm{div}(\vec{F}) = \nabla \cdot \vec{F}$. Formulating formulas using the "Nabla vector" and using rules from geometry is called **Nabla calculus**. This works both in 2 and 3 dimensions even so the ∇ vector is not an actual vector but an operator. The following combination of divergence and gradient often appears in physics:

$$\Delta f = \mathrm{div}(\mathrm{grad}(f)) = f_{xx} + f_{yy} + f_{zz} \; .$$

It is called the Laplacian of f. We can write $\Delta f = \nabla^2 f$ because $\nabla \cdot (\nabla f) = \mathrm{div}(\mathrm{grad}(f)$.

We can extend the Laplacian also to vector fields with

$\Delta \vec{F} = (\Delta P, \Delta Q, \Delta R)$ and write $\nabla^2 \vec{F}$.

Here are some identities:

$\mathrm{div}(\mathrm{curl}(\vec{F})) = 0$.
$\mathrm{curlgrad}(\vec{F}) = \vec{0}$
$\mathrm{curl}(\mathrm{curl}(\vec{F})) = \mathrm{grad}(\mathrm{div}(\vec{F}) - \Delta(\vec{F}))$.

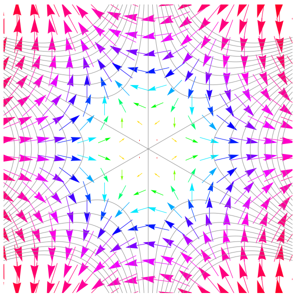
Proof. $\nabla \cdot \nabla \times \vec{F} = 0$.
$\nabla \times \nabla \vec{F} = \vec{0}$.
$\nabla \times \nabla \times \vec{F} = \nabla(\nabla \cdot \vec{F}) - (\nabla \cdot \nabla)\vec{F}$.

1 **Question:** Is there a vector field $\vec{G}$ such that $\vec{F} = \langle x + y, z, y^2 \rangle = \mathrm{curl}(\vec{G})$?
Answer: No, because $\mathrm{div}(\vec{F}) = 1$ is incompatible with $\mathrm{div}(\mathrm{curl}(\vec{G})) = 0$.

2 Show that in simply connected region, every irrotational and incompressible field can be written as a vector field $\vec{F} = \mathrm{grad}(f)$ with $\Delta f = 0$. Proof. Since $\vec{F}$ is irrotational, there exists a function f satisfying $F = \mathrm{grad}(f)$. Now, $\mathrm{div}(F) = 0$ implies $\mathrm{divgrad}(f) = \Delta f = 0$.

3 Find an example of a field which is both incompressible and irrotational. Solution. Find f which satisfies the Laplace equation $\Delta f = 0$, like $f(x,y) = x^3 - 3xy^2$, then look at its gradient field $\vec{F} = \nabla f$. In that case, this gives

$$\vec{F}(x,y) = \langle 3x^2 - 3y^2, -6xy \rangle \; .$$



4 If we rotate the vector field $\vec{F} = \langle P, Q \rangle$ by 90 degrees $= \pi/2$, we get a new vector field $\vec{G} = \langle -Q, P \rangle$. The integral $\int_C F \ \cdot ds$ becomes a **flux** $\int_\gamma G \cdot dn$ of G through the boundary of R, where dn is a normal vector with length $|r'|dt$. With $\text{div}(\vec{F}) = (P_x + Q_y)$, we see that

$$\text{curl}(\vec{F}) = \text{div}(\vec{G}) \ .$$

Green's theorem now becomes

$$\int\int_R \text{div}(\vec{G}) \ dxdy = \int_C \vec{G} \cdot \vec{dn} \ ,$$

where $dn(x, y)$ is a normal vector at (x, y) orthogonal to the velocity vector $\vec{r}\,'(x, y)$ at (x, y). This new theorem has a generalization to three dimensions, where it is called Gauss theorem or divergence theorem. Don't treat this however as a different theorem in two dimensions. It is just Green's theorem in disguise.

This result shows:

> The divergence at a point (x, y) is the average flux of the field through a small circle of radius r around the point in the limit when the radius of the circle goes to zero.

We have now all the derivatives we need. In dimension d, there are d fundamental derivatives.

$$1$$

$$1 \xrightarrow{\text{grad}} 1$$

$$1 \xrightarrow{\text{grad}} 2 \xrightarrow{\text{curl}} 1$$

$$1 \xrightarrow{\text{grad}} 3 \xrightarrow{\text{curl}} 3 \xrightarrow{\text{div}} 1$$

They are all part of an **exterior derivative**. To the end, let me stress that it is important you keep the dimensions. Many books treat two dimensional situations using terminology from three dimensions which leads to confusion. Geometry in two dimensions should be treated as a "flatlander"[1] and use two dimensions only. It is a modern point of view that geometry should be done with intrinsic notions which do not assume that geometry is part of a larger space. Integral theorems become more transparent if you look at them in the right dimension. In one dimension, we had one theorem, the fundamental theorem of calculus. In two dimensions, there is the fundamental theorem of line integrals and Greens theorem. In three dimensions there are three theorems: the fundamental theorem of line integrals, Stokes theorem and the divergence theorem. We will look at the remaining two theorems in the next class.

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[1] A. Abbott, Flatland, A romance in many dimensions,1884

Homework

1 Find your own nonzero vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ in each of the following cases:
a) $\vec{F}$ is irrotational but not incompressible.
b) $\vec{F}$ is incompressible but not irrotational.
c) $\vec{F}$ is irrotational and incompressible.
d) $\vec{F}$ is not irrotational and not incompressible.

2 The vector field $\vec{F}(x,y,z) = \langle x, y, -2z\rangle$ satisfies $\mathrm{div}(\vec{F}) = 0$. Can you find a vector field $\vec{G}(x,y,z)$ such that $\mathrm{curl}(\vec{G}) = \vec{F}$? Such a field $\vec{G}$ is called a **vector potential**.
Hint. Write $\vec{F}$ as a sum $\langle x, 0, -z\rangle + \langle 0, y, -z\rangle$ and find vector potentials for each of the parts using a vector field you have seen on the blackboard in class.

3 Evaluate the flux integral $\int\int_S \langle 0, 0, yz\rangle \cdot d\vec{S}$, where S is the surface with parametric equation $x = uv, y = u + v, z = u - v$ on $R : u^2 + v^2 \leq 4$.

4 Evaluate the flux integral $\int_S \mathrm{curl}(F) \cdot d\vec{S}$ for $\vec{F}(x,y,z) = \langle xy, yz, zx\rangle$, where S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $[0,1] \times [0,1]$ and has an upward orientation.

5 a) What is the relation between the flux of the vector field $\vec{F} = \nabla g / |\nabla g|$ through the surface $S : \{g = 1\}$ with $g(x,y,z) = x^6 + y^4 + 2z^8$ and the surface area of S?
b) Find the flux of the vector field $\vec{G} = \nabla g \times \langle 0, 0, 1\rangle$ through the surface S.

Remark This problem, both part a) and part do not need any computation. You can answer each question with one sentence. In part a) compare $\vec{F} \cdot d\vec{S}$ with dS in that case.

Lecture 23: Stokes Theorem

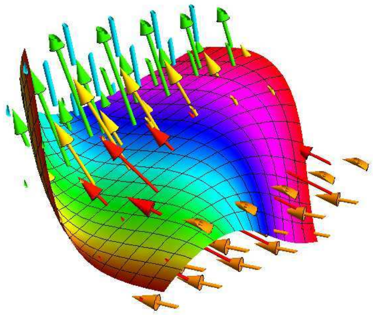
Assume a surface S is parametrized as $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ over a domain G in the uv-plane.

The **flux integral** of $\vec{F}$ through S is defined as the double integral

$$\int\int_G \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\; dudv\ .$$

One uses the short hand notation $\vec{dS} = (\vec{r}_u \times \vec{r}_v)\; dudv$ and thinks about $\vec{dS}$ as an infinitesimal normal vector to the surface. The flux integral can be abbreviated as $\int\int_S \vec{F}\cdot\vec{dS}$. The interpretation is that if $\vec{F}$ = fluid velocity field, then $\int\int_S \vec{F}\cdot\vec{dS}$ is the amount of fluid passing through S in unit time.

Because $\vec{n} = \vec{r}_u \times \vec{r}_v / |\vec{r}_u \times \vec{r}_v|$ is a unit vector normal to the surface and on the surface, $\vec{F}\cdot\vec{n}$ is the normal component of the vector field with respect to the surface. One could write therefore also $\int\int_S \vec{F}\cdot\vec{dS} = \int\int \vec{F}\cdot\vec{n}\; dS$, where dS is the surface element we know from when we computed surface area. The function $\vec{F}\cdot\vec{n}$ is the scalar projection of $\vec{F}$ in the normal direction. Whereas the formula $\int\int 1\; dS$ gave the area of the surface with $dS = |\vec{r}_u \times \vec{r}_v| dudv$, the flux integral weights each area element dS with the normal component of the vector field with $\vec{F}(\vec{r}(u,v) \cdot \vec{n}(\vec{r}(u,v))$. It is important that we do not want to use this formula for computations (even so it appears in books) because finding $\vec{n}$ gives additional work. We just determine the vectors $\vec{F}(\vec{r}(u,v))$ and $\vec{r}_u \times \vec{r}_v$ and integrate its dot product over the domain G.



1 Compute the flux of $\vec{F}(x,y,z) = \langle 0, 1, z^2\rangle$ through the upper half sphere S parametrized by

$$\vec{r}(u,v) = \langle \cos(u)\sin(v), \sin(u)\sin(v), \cos(v)\rangle\ .$$

Solution. We have $\vec{r}_u \times \vec{r}_v = -\sin(v)\vec{r}$ and $\vec{F}(\vec{r}(u,v)) = \langle 0, 1, \cos^2(v)\rangle$ so that

$$\int_0^{2\pi}\int_0^{\pi} -\langle 0, 1, \cos^2(v)\rangle \cdot \langle \cos(u)\sin^2(v), \sin(u)\sin^2(v), \cos(v)\sin(v)\rangle\; dudv\ .$$

The flux integral is $\int_0^{2\pi}\int_{\pi/2}^{\pi} -\sin^2(v)\sin(u) - \cos^3(v)\sin(v)\; dudv$ which is $-\int_{\pi/2}^{\pi}\cos^3 v\sin(v)\; dv = \cos^4(v)/4|_0^{\pi/2} = -1/4$.

2 Calculate the flux of the vector field $\vec{F}(x,y,z) = \langle 1,2,4z \rangle$ through the paraboloid $z = x^2 + y^2$ lying above the region $x^2 + y^2 \leq 1$. **Solution:** We can parametrize the surface as $\vec{r}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), r^2 \rangle$ where $\vec{r}_r \times \vec{r}_\theta = \langle -2r^2\cos(\theta), -2r^2\sin(\theta), r \rangle$ and $\vec{F}(\vec{r}(u,v)) = \langle 1,2,4r^2 \rangle$. We get $\int_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^1 (-2r^2\cos(v) - 4r^2\sin(v) + 4r^3)\, drd\theta = 2\pi$.

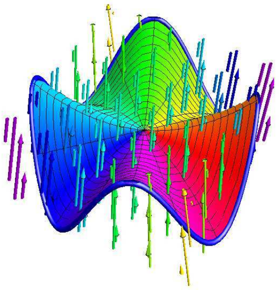
3 Compute the flux of $\vec{F}(x,y,z) = \langle 2,3,1 \rangle$ through the torus parameterized as $\vec{r}(u,v) = \langle (2+\cos(v))\cos(u), (2+\cos(v))\sin(u), \sin(v) \rangle$, where both u and v range from 0 to $2pi$. **Solution.** There is no computation is needed. Think about what the flux means.

The following theorem is the second fundamental theorem of calculus in three dimensions:

The **boundary** of a surface S consists of all points P where even arbitrary small circle $S_r(P) \cap S$ around the point is not closed. It is a curve oriented so that the surface is to the "left" if the normal vector to the surface is pointing "up". In other words, the velocity vector v, a vector w pointing towards the surface and the normal vector n to the surface form a right handed coordinate system.

Stokes theorem: Let S be a surface bounded by a curve C and $\vec{F}$ be a vector field. Then

$$\int\int_S \operatorname{curl}(\vec{F}) \ \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} \ .$$



Proof. Stokes theorem can be proven in the same way than Greens theorem. Chop up S as a union of small triangles. As before, the sum of the fluxes through all these triangles adds up to the flux through the surface and the sum of the line integrals along the boundaries adds up to the line integral of the boundary of S. Stokes theorem for a small triangle can be reduced to Greens theorem because with a coordinate system such that the triangle is in the xy plane, the flux of the field is the double integral of $\operatorname{curl}\vec{F}\ d\vec{S} = \operatorname{curl}F(\vec{r}) \cdot \vec{n} dudv = (Q_x - P_y)\cos(\theta) dudv$ where θ is the angle between the normal vector and $\vec{F} = \langle P,Q,R \rangle$. On the other hand, since the power $\vec{F}(\vec{r}) \cdot \vec{r}'(t)dt = (P(\vec{r})\cos(\theta)x'(t) + Q(\vec{r})\cos(\theta)y'(t))dt$ also has everything multiplied by $\cos(\theta)$, the result for the space triangle follows from Green.

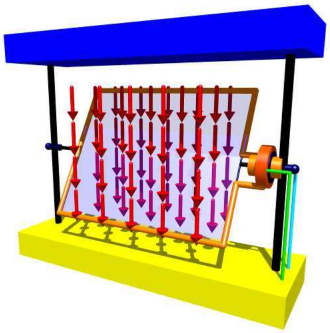
4 Let $\vec{F}(x,y,z) = \langle -y,x,0 \rangle$ and let S be the upper semi hemisphere, then $\operatorname{curl}(\vec{F})(x,y,z) = \langle 0,0,2 \rangle$. The surface is parameterized by $\vec{r}(u,v) = \langle \cos(u)\sin(v), \sin(u)\sin(v), \cos(v) \rangle$ on $G = [0,2\pi] \times [0,\pi/2]$ and $\vec{r}_u \times \vec{r}_v = \sin(v)\vec{r}(u,v)$ so that $\operatorname{curl}(\vec{F})(x,y,z) \cdot \vec{r}_u \times \vec{r}_v = \cos(v)\sin(v)2$. The integral $\int_0^{2\pi} \int_0^{\pi/2} \sin(2v)\ dvdu = 2\pi$.
The boundary C of S is parameterized by $\vec{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$ so that $d\vec{r} = \vec{r}\,'(t)dt = \langle -\sin(t), \cos(t), 0 \rangle\ dt$ and $\vec{F}(\vec{r}(t))\ \vec{r}\,'(t)dt = \sin(t)^2 + \cos^2(t) = 1$. The line integral $\int_C \vec{F} \cdot d\vec{r}$ along the boundary is 2π.

5 If S is a surface in the xy-plane and $\vec{F} = \langle P, Q, 0 \rangle$ has zero z component, then $\text{curl}(\vec{F}) = \langle 0, 0, Q_x - P_y \rangle$ and $\text{curl}(\vec{F}) \cdot \vec{dS} = Q_x - P_y \; dxdy$. We see that for a surface which is flat, Stokes theorem is a consequence of Green's theorem. If we put the coordinate axis so that the surface is in the xy-plane, then the vector field F induces a vector field on the surface such that its $2D$ curl is the normal component of $\text{curl}(F)$. The reason is that the third component $Q_x - P_y$ of $\text{curl}(\vec{F})\langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$ is the two dimensional curl: $\vec{F}(\vec{r}(u,v)) \cdot \langle 0, 0, 1 \rangle = Q_x - P_y$. If C is the boundary of the surface, then $\int\int_S \vec{F}(\vec{r}(u,v)) \cdot \langle 0, 0, 1 \rangle \; dudv = \int_C \vec{F}(\vec{r}(t)) \; \vec{r}\,'(t) dt$.

6 Calculate the flux of the curl of $\vec{F}(x,y,z) = \langle -y, x, 0 \rangle$ through the surface parameterized by $\vec{r}(u,v) = \langle \cos(u)\cos(v), \sin(u)\cos(v), \cos^2(v) + \cos(v)\sin^2(u + \pi/2) \rangle$. Because the surface has the same boundary as the upper half sphere, the integral is again 2π as in the above example.

7 For every surface bounded by a curve C, the flux of $\text{curl}(\vec{F})$ through the surface is the same. Proof. The flux of the curl of a vector field through a surface S depends only on the boundary of S. Compare this with the earlier statement that for every curve between two points A, B the line integral of $\text{grad}(f)$ along C is the same. The line integral of the gradient of a function of a curve C depends only on the end points of C.

8 Electric and magnetic fields are linked by the **Maxwell equation** $\text{curl}(\vec{E}) = -\frac{1}{c}\dot{B}$. If a closed wire C bounds a surface S then $\int\int_S B \cdot dS$ is the flux of the magnetic field through S. Its change can be related with a voltage using Stokes theorem: $d/dt \int\int_S B \cdot dS = \int\int_S \dot{B} \cdot dS = \int\int_S -c \; \text{curl}(\vec{E}) \cdot \vec{dS} = -c \int_C \vec{E} \; \vec{dr} = U$, where U is the voltage. If we change the flux of the magnetic field through the wire, then this induces a voltage. The flux can be changed by changing the amount of the magnetic field but also by changing the direction. If we turn around a magnet around the wire or the wire inside the magnet, we get an electric voltage. This happens in a power-generator like an alternator in a car. Stokes theorem explains why we can generate electricity from motion.



Stokes theorem was found by Ampère in 1825. George Gabriel Stokes (1819-1903) (probably inspired by work of Green) rediscovers the identity around 1840.

George Gabriel Stokes

André Marie Ampere

Homework

1 Find $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y,z) = \langle 2x^2y, 2x^3/3, 2xy \rangle$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above.

2 If S is the surface $x^8 + y^8 + z^6 = 1$ and assume $\vec{F}$ is a smooth vector field in space. Explain why $\int_S \mathrm{curl}(\vec{F}) \cdot d\vec{S} = 0$.

3 Evaluate the flux integral $\int\int_S \mathrm{curl}(\vec{F}) \cdot d\vec{S}$, where

$$\vec{F}(x,y,z) = \langle xe^{y^2}z^3 + 2xyze^{x^2+z}, x + z^2e^{x^2+z}, ye^{x^2+z} + ze^x \rangle$$

and where S is the part of the ellipsoid $x^2 + y^2/4 + (z+1)^2 = 2$, $z > 0$ oriented so that the normal vector points upwards.

4 Find the line integral $\int_C \vec{F}\ d\vec{r}$, where C is the circle of radius 3 in the xz-plane oriented counter clockwise when looking from the point $(0,1,0)$ onto the plane and where $\vec{F}$ is the vector field

$$\vec{F}(x,y,z) = \langle 2x^2z + x^5, \cos(e^y), -2xz^2 + \sin(\sin(z)) \rangle \ .$$

Use a convenient surface S which has C as a boundary.

5 Find the flux integral $\int_S \mathrm{curl}(\vec{F}) \cdot d\vec{S}$, where $\vec{F}(x,y,z) =$

$$\langle 2\cos(\pi y)e^{2x} + z^2, x^2\cos(z\pi/2) - \pi\sin(\pi y)e^{2x}, 2xz \rangle$$

and S is the surface parametrized by

$$\vec{r}(s,t) = \langle (1 - s^{1/3})\cos(t) - 4s^2, (1 - s^{1/3})\sin(t), 5s \rangle$$

with $0 \le t \le 2\pi, 0 \le s \le 1$ and oriented so that the normal vectors point to the outside of the thorn.

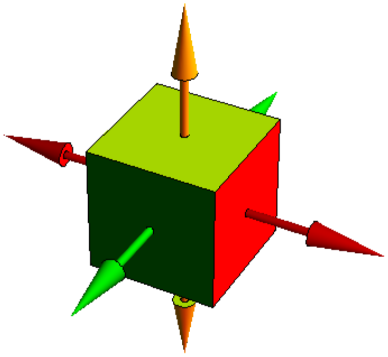
Lecture 24: Divergence theorem

There are three integral theorems in three dimensions. We have seen already the fundamental theorem of line integrals and Stokes theorem. Here is the divergence theorem, which completes the list of integral theorems in three dimensions:

Divergence Theorem. Let E be a solid with boundary surface S oriented so that the normal vector points outside. Let $\vec{F}$ be a vector field. Then

$$\int\int\int_E \mathrm{div}(\vec{F})\, dV = \int\int_S \vec{F} \cdot dS \,.$$

To prove this, one can look at a small box $[x, x+dx] \times [y, y+dy] \times [z, z+dz]$. The flux of $\vec{F} = \langle P, Q, R \rangle$ through the faces perpendicular to the x-axes is $[\vec{F}(x+dx, y, z) \cdot \langle 1, 0, 0 \rangle + \vec{F}(x, y, z) \cdot \langle -1, 0, 0 \rangle] dydz = P(x+dx, y, z) - P(x, y, z) = P_x\, dxdydz$. Similarly, the flux through the y-boundaries is $P_y\ dydxdz$ and the flux through the two z-boundaries is $P_z\ dzdxdy$. The total flux through the faces of the cube is $(P_x + P_y + P_z)\ dxdydz = \mathrm{div}(\vec{F})\ dxdydz$. A general solid can be approximated as a union of small cubes. The sum of the fluxes through all the cubes consists now of the flux through all faces without neighboring faces. and fluxes through adjacent sides cancel. The sum of all the fluxes of the cubes is the flux through the boundary of the union. The sum of all the $\mathrm{div}(\vec{F})\ dxdydz$ is a Riemann sum approximation for the integral $\int\int\int_G \mathrm{div}(\vec{F})\ dxdydz$. In the limit, where dx, dy, dz goes to zero, we obtain the divergence theorem.



The theorem explains what divergence means. If we average the divergence over a small cube is equal the flux of the field through the boundary of the cube. If this is positive, then more field exists the cube than entering the cube. There is field "generated" inside. The divergence measures the expansion of the field.

1 Let $\vec{F}(x, y, z) = \langle x, y, z \rangle$ and let S be sphere. The divergence of $\vec{F}$ is the constant function $\text{div}(\vec{F}) = 3$ and $\int\int\int_G \text{div}(\vec{F})\ dV = 3 \cdot 4\pi/3 = 4\pi$. The flux through the boundary is $\int\int_S \vec{r} \cdot \vec{r}_u \times \vec{r}_v\ dudv = \int\int_S |\vec{r}(u, v)|^2 \sin(v)\ dudv = \int_0^\pi \int_0^{2\pi} \sin(v)\ dudv = 4\pi$ also. We see that the divergence theorem allows us to compute the area of the sphere from the volume of the enclosed ball or compute the volume from the surface area.

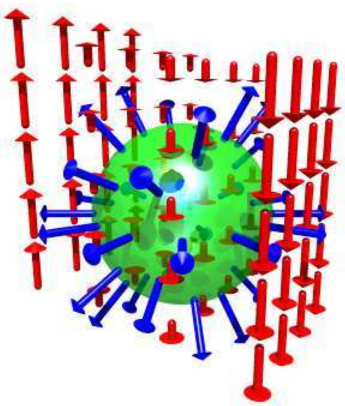
2 What is the flux of the vector field $\vec{F}(x, y, z) = \langle 2x, 3z^2 + y, \sin(x) \rangle$ through the solid $G = [0, 3] \times [0, 3] \times [0, 3] \setminus ([0, 3] \times [1, 2] \times [1, 2] \cup [1, 2] \times [0, 3] \times [1, 2] \cup [0, 3] \times [0, 3] \times [1, 2])$ which is a cube where three perpendicular cubic holes have been removed? **Solution:** Use the divergence theorem: $\text{div}(\vec{F}) = 2$ and so $\int\int\int_G \text{div}(\vec{F})\ dV = 2\int\int\int_G\ dV = 2\text{Vol}(G) = 2(27 - 7) = 40$. Note that the flux integral here would be over a complicated surface over dozens of rectangular planar regions.

3 Find the flux of $\text{curl}(F)$ through a torus if $\vec{F} = \langle yz^2, z + \sin(x) + y, \cos(x) \rangle$ and the torus has the parametrization

$$\vec{r}(\theta, \phi) = \langle (2 + \cos(\phi)) \cos(\theta), (2 + \cos(\phi)) \sin(\theta), \sin(\phi) \rangle \ .$$

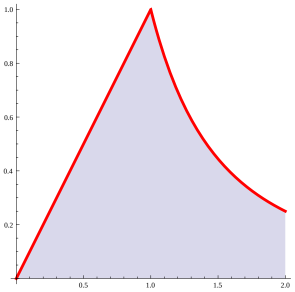
Solution: The answer is 0 because the divergence of $\text{curl}(F)$ is zero. By the divergence theorem, the flux is zero.

4 Similarly as Green's theorem allowed to calculate the area of a region by passing along the boundary, the volume of a region can be computed as a flux integral: Take for example the vector field $\vec{F}(x, y, z) = \langle x, 0, 0 \rangle$ which has divergence 1. The flux of this vector field through the boundary of a solid region is equal to the volume of the solid: $\int\int_{\delta G} \langle x, 0, 0 \rangle \cdot \vec{dS} = \text{Vol}(G)$.



5 How heavy are we, at distance r from the center of the earth?
Solution: The law of gravity can be formulated as $\text{div}(\vec{F}) = 4\pi\rho$, where ρ is the mass density. We assume that the earth is a ball of radius R. By rotational symmetry, the gravitational force is normal to the surface: $\vec{F}(\vec{x}) = \vec{F}(r)\vec{x}/||\vec{x}||$. The flux of $\vec{F}$ through a ball of radius r is $\int\int_{S_r} \vec{F}(x) \cdot \vec{dS} = 4\pi r^2 \vec{F}(r)$. By the **divergence theorem**, this is

$4\pi M_r = 4\pi \int \int \int_{B_r} \rho(x)\ dV$, where M_r is the mass of the material inside S_r. We have $(4\pi)^2 \rho r^3/3 = 4\pi r^2 \vec{F}(r)$ for $r < R$ and $(4\pi)^2 \rho R^3/3 = 4\pi r^2 \vec{F}(r)$ for $r \geq R$. Inside the earth, the gravitational force $\vec{F}(r) = 4\pi \rho r/3$. Outside the earth, it satisfies $\vec{F}(r) = M/r^2$ with $M = 4\pi R^3 \rho/3$.



To the end we make an overview over the integral theorems and give an other typical example in each case.

The fundamental theorem for line integrals, Green's theorem, Stokes theorem and divergence theorem are all incarnation of **one single theorem** $\int_A dF = \int_{\delta A} F$, where dF is a **exterior derivative** of F and where δA is the **boundary** of A. They all generalize the fundamental theorem of calculus.

Fundamental theorem of line integrals: If C is a curve with boundary $\{A, B\}$ and f is a function, then

$$\int_C \nabla f \cdot \vec{dr} = f(B) - f(A)$$

Remarks.

1) For closed curves, the line integral $\int_C \nabla f \cdot \vec{dr}$ is zero.
2) Gradient fields are **path independent**: if $\vec{F} = \nabla f$, then the line integral between two points P and Q does not depend on the path connecting the two points.
3) The theorem holds in any dimension. In one dimension, it reduces to the **fundamental theorem of calculus** $\int_a^b f'(x)\ dx = f(b) - f(a)$
4) The theorem justifies the name **conservative** for gradient vector fields.
5) The term "potential" was coined by George Green who lived from 1783-1841.
Example. Let $f(x, y, z) = x^2 + y^4 + z$. Find the line integral of the vector field $\vec{F}(x, y, z) = \nabla f(x, y, z)$ along the path $\vec{r}(t) = \langle \cos(5t), \sin(2t), t^2 \rangle$ from $t = 0$ to $t = 2\pi$.

Solution. $\vec{r}(0) = \langle 1, 0, 0 \rangle$ and $\vec{r}(2\pi) = \langle 1, 0, 4\pi^2 \rangle$ and $f(\vec{r}(0)) = 1$ and $f(\vec{r}(2\pi)) = 1 + 4\pi^2$. The fundamental theorem of line integral gives $\int_C \nabla f\ \vec{dr} = f(r(2\pi)) - f(r(0)) = 4\pi^2$.

Green's theorem. If R is a region with boundary C and $\vec{F}$ is a vector field, then

$$\int \int_R \text{curl}(\vec{F})\ dxdy = \int_C \vec{F} \cdot \vec{dr}\ .$$

Remarks.
1) Greens theorem allows to switch from double integrals to one dimensional integrals.
2) The curve is oriented in such a way that the region is to the left.
3) The boundary of the curve can consist of piecewise smooth pieces.
4) If $C : t \mapsto \vec{r}(t) = \langle x(t), y(t) \rangle$, the line integral is $\int_a^b \langle P(x(t), y(t)), Q(x(t), y(t)) \rangle \cdot \langle x'(t), y'(t) \rangle \, dt$.
5) Green's theorem was found by George Green (1793-1841) in 1827 and by Mikhail Ostrogradski (1801-1862).
6) If $\text{curl}(\vec{F}) = 0$ in a simply connected region, then the line integral along a closed curve is zero. If two curves connect two points then the line integral along those curves agrees.
7) Taking $\vec{F}(x, y) = \langle -y, 0 \rangle$ or $\vec{F}(x, y) = \langle 0, x \rangle$ gives **area formulas**.

Example. Find the line integral of the vector field $\vec{F}(x, y) = \langle x^4 + \sin(x) + y, x + y^3 \rangle$ along the path $\vec{r}(t) = \langle \cos(t), 5\sin(t) + \log(1 + \sin(t)) \rangle$, where t runs from $t = 0$ to $t = \pi$.

Solution. $\text{curl}(\vec{F}) = 0$ implies that the line integral depends only on the end points $(0, 1), (0, -1)$ of the path. Take the simpler path $\vec{r}(t) = \langle -t, 0 \rangle, -1 \leq t \leq 1$, which has velocity $\vec{r}\,'(t) = \langle -1, 0 \rangle$. The line integral is $\int_{-1}^{1} \langle t^4 - \sin(t), -t \rangle \cdot \langle -1, 0 \rangle \, dt = -t^5/5|_{-1}^{1} = -2/5$.

Remark We could also find a potential $f(x, y) = x^5/5 - \cos(x) + xy + y^5/4$. It has the property that $\text{grad}(f) = F$. Again, we get $f(0, -1) - f(0, 1) = -1/5 - 1/5 = -2/5$.

Stokes theorem. If S is a surface with boundary C and $\vec{F}$ is a vector field, then

$$\int\int_S \text{curl}(\vec{F}) \cdot dS = \int_C \vec{F} \cdot \vec{dr} \, .$$

Remarks.
1) Stokes theorem allows to derive Greens theorem: if $\vec{F}$ is z-independent and the surface S is contained in the xy-plane, one obtains the result of Green.
2) The orientation of C is such that if you walk along C and have your head in the direction of the normal vector $\vec{r}_u \times \vec{r}_v$, then the surface to your left.
3) Stokes theorem was found by André Ampère (1775-1836) in 1825 and rediscovered by George Stokes (1819-1903).
4) The flux of the curl of a vector field does not depend on the surface S, only on the boundary of S.
5) The flux of the curl through a closed surface like the sphere is zero: the boundary of such a surface is empty.

Example. Compute the line integral of $\vec{F}(x, y, z) = \langle x^3 + xy, y, z \rangle$ along the polygonal path C connecting the points $(0, 0, 0), (2, 0, 0), (2, 1, 0), (0, 1, 0)$.
Solution. The path C bounds a surface $S : \vec{r}(u, v) = \langle u, v, 0 \rangle$ parameterized by $R = [0, 2] \times [0, 1]$. By Stokes theorem, the line integral is equal to the flux of $\text{curl}(\vec{F})(x, y, z) = \langle 0, 0, -x \rangle$ through S. The normal vector of S is $\vec{r}_u \times \vec{r}_v = \langle 1, 0, 0 \rangle \times \langle 0, 1, 0 \rangle = \langle 0, 0, 1 \rangle$ so that $\int\int_S \text{curl}(\vec{F}) \, \vec{dS} = \int_0^2 \int_0^1 \langle 0, 0, -u \rangle \cdot \langle 0, 0, 1 \rangle \, dudv = \int_0^2 \int_0^1 -u \, dudv = -2$.

Divergence theorem: If S is the boundary of a region E in space and $\vec{F}$ is a vector field, then

$$\int\int\int_B \text{div}(\vec{F}) \, dV = \int\int_S \vec{F} \cdot \vec{dS} \, .$$

Remarks.
1) The divergence theorem is also called **Gauss theorem**.
2) It can be helpful to determine the flux of vector fields through surfaces.
3) It was discovered in 1764 by Joseph Louis Lagrange (1736-1813), later it was rediscovered by Carl Friedrich Gauss (1777-1855) and by George Green.
4) For divergence free vector fields $\vec{F}$, the flux through a closed surface is zero. Such fields $\vec{F}$ are also called **incompressible** or **source free**.

Example. Compute the flux of the vector field $\vec{F}(x, y, z) = \langle -x, y, z^2 \rangle$ through the boundary S of the rectangular box $[0, 3] \times [-1, 2] \times [1, 2]$.
Solution. By Gauss theorem, the flux is equal to the triple integral of $\text{div}(F) = 2z$ over the box: $\int_0^3 \int_{-1}^2 \int_1^2 2z \, dxdydz = (3 - 0)(2 - (-1))(4 - 1) = 27$.

How do these theorems fit together? In n-dimensions, there are n theorems. We have here seen the situation in dimension n=2 and n=3, but one could continue. The fundamental theorem of line integrals generalizes directly to higher dimensions. Also the divergence theorem generalizes directly since an n-dimensional integral in n dimensions. The generalization of curl and flux is more subtle, since in 4 dimensions already, the curl of a vector field is a 6 dimensional object. It is a $n(n-1)/2$ dimensional object in general.

$$\begin{tikzcd}
 & & & 1 & & & \\
 & & 1 \arrow[rr, "\text{FTC}"] & & 1 & & \\
 & 1 \arrow[rr, "\text{FTL}"] & & 2 \arrow[rr, "\text{Green}"] & & 1 & \\
1 \arrow[rr, "\text{FTL}"] & & 3 \arrow[rr, "\text{Stokes}"] & & 3 \arrow[rr, "\text{Gauss}"] & & 1
\end{tikzcd}$$

In one dimensions, there is one derivative $f(x) \to f'(x)$ from scalar to scalar functions. It corresponds to the entry $1 - 1$ in the Pascal triangle. The next entry $1 - 2 - 1$ corresponds to differentiation in two dimensions, where we have the gradient $f \to \nabla f$ mapping a scalar function to a vector field with 2 components as well as the curl, $F \to \text{curl}(F)$ which corresponds to the transition $2 - 1$. The situation in three dimensions is captured by the entry $1 - 3 - 3 - 1$ in the Pascal triangle. The first derivative $1 - 3$ is the gradient. The second derivative $3 - 3$ is the curl and the third derivative $3 - 1$ is the divergence. In $n = 4$ dimensions, we would have to look at $1 - 4 - 6 - 4 - 1$. The first derivative $1 - 4$ is still the gradient. Then we have a first curl, which maps a vector field with 4 components into an object with 6 components. Then there is a second curl, which maps an object with 6 components back to a vector field, we would have to look at $1 - 4 - 6 - 4 - 1$. When setting up calculus in dimension n, one talks about **differential forms** instead of scalar fields or vector fields. Functions are 0 forms or n-forms. Vector fields can be described by 1 or $n - 1$ forms. The general formalism defines a derivative d called **exterior derivative** on differential forms as well as integration of such k forms on k dimensional objects. There is a **boundary operation** δ which maps a k-dimensional object into a $k - 1$ dimensional object. This boundary operation is dual to differentiation. They both satisfy the same relation $dd(F) = 0$ and $\delta\delta G = 0$. Differentiation and integration are linked by the general Stokes theorem:

$$\int_{\delta G} F = \int_G dF$$

which becomes a single theorem called **fundamental theorem of multivariable calculus**. The theorem becomes much simpler in quantum calculus, where geometric objects and differential forms are on the same footing. It tuns out that the theorem becomes then $< \delta G, F >=< G, dF >$ which you might see in linear algebra in the form $< A^T v, w >=< v, Aw >$, where A is a matrix and $< v, w >$ is the dot product. If we deal with "smooth" functions and fields that we have to pay a prize and consider in turn "singular" objects like points or curves and surfaces. These are idealized objects which have zero diameter, radius or thickness. Nature likes simplicity and elegance [1]: and has chosen quantum mathematics to be more fundamental but it manifests only in the very small. While it is well understood mathematically, it will take a while until this formalism will enter calculus courses.

Homework

1 Compute using the divergence theorem the flux of the vector field $\vec{F}(x,y,z) = \langle 3y, xy, 2yz \rangle$ through the unit cube $[0,1] \times [0,1] \times [0,1]$.

2 Find the flux of the vector field $\vec{F}(x,y,z) = \langle xy, yz, zx \rangle$ through the solid cylinder $x^2 + y^2 \leq 1$, $0 \leq z \leq 1$.

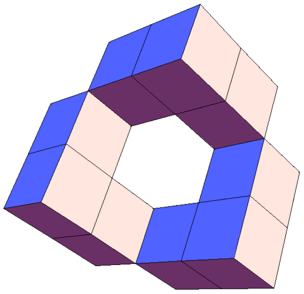
3 Use the divergence theorem to calculate the flux of $\vec{F}(x,y,z) = \langle x^3, y^3, z^3 \rangle$ through the sphere $S : x^2 + y^2 + z^2 = 1$ where the sphere is oriented so that the normal vector points outwards.

4 Assume the vector field

$$\vec{F}(x,y,z) = \langle 5x^3 + 12xy^2, y^3 + e^y \sin(z), 5z^3 + e^y \cos(z) \rangle$$

is the magnetic field of the **sun** whose surface is a sphere of radius 3 oriented with the outward orientation. Compute the magnetic flux $\int_S \vec{F} \cdot d\vec{S}$.

5 Find $\int\int_S \vec{F} \cdot d\vec{S}$, where $\vec{F}(x,y,z) = \langle x, y, z \rangle$ and S is the boundary of the solid built with 9 unit cubes shown in the picture.



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[1] Leibniz: 1646-1716