8/8/2013 PRACTICE EXAM III **Maths 21a, O. Knill, Summer 2013**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points)

1) T **F** The line $\vec{r}(t) = \langle t, t, -t \rangle$ is is contained in the plane $x + y + z = 1$.

Solution:
Already for $t = 0$ we are not on the plane.

2) T **F** The quadratic surface $x^2 + y^2 = z^2$ is an elliptic paraboloid.

Solution:
It is a cone.

3) **T** F If $\vec{T}(t), \vec{B}(t), \vec{N}(t)$ are the unit tangent, normal and binormal vectors of a curve with $\vec{r}\,'(t) \neq 0$ everywhere, then $\vec{T}(t) \cdot \vec{B}(t) \times \vec{N}(t)$ is always equal to 1 or -1.

Solution:
The three vectors are perpendicular and have length 1.

4) **T** F If $|\vec{u} \times \vec{v}| = 0$, then $\text{Proj}_{\vec{u}}(\vec{v}) = \vec{u}$.

Solution:
The two vectors are then parallel.

5) **T** F There is a vector field $\vec{F}(x, y)$ which has the property $\text{curl}(\vec{F}) = \text{div}(\vec{F}) = 1$. where $\text{curl}(\vec{F})(x, y) = Q_x(x, y) - P_y(x, y)$ and $\text{div}(\vec{F})(x, y) = P_x(x, y) + Q_y(x, y)$.

Solution:
$\vec{F}(x, y) = \langle x, x \rangle$ is an example.

6) T **F** The acceleration vector $\vec{r}\,''(t) = \langle x(t), y(t) \rangle$ is always in the plane spanned by the vector $\vec{r}(t)$ and the velocity vector $\vec{r}\,'(t)$.

Solution:
Take $\vec{r}(t) = \langle \cos(t), \sin(t), 1\rangle$.

7) T **F** For every curve on the unit sphere, the curvature is constant and equal to 1.

Solution:
Draw a small circle on the sphere.

8) T **F** If a smooth function $f(x,y)$ has no maximum nor minimum, then it does not have a critical point.

Solution:
It can have a saddle point.

9) T **F** The linearization $L(x,y)$ of a cubic function $f(x,y) = x^3+y^3$ is the function $L(x,y) = 3x^2 + 3y^2$.

Solution:
The linearization is a linear function.

10) **T** F If $\vec{F}(x,y)$ is a gradient field $\vec{F} = \nabla f$ and $\vec{r}(t)$ is a flow line satisfying $\vec{r}\,'(t) = \vec{F}(\vec{r}(t))$ then $\frac{d}{dt} f(\vec{r}(t)) = |F|^2(\vec{r}(t))$.

Solution:
This is a consequence of the chain rule.

11) **T** F If $f+g$ and $f-g$ have a common critical point (a,b), then this point is a critical point of both f and g.

Solution:
$\nabla f + \nabla g = 0$ and $\nabla f - \nabla g = 0$ implies $\nabla f = \nabla g = 0$.

12) **T** F Assume a vector field $\vec{F}(x,y,z)$ is a gradient field, then $\int_C \vec{F} \cdot \vec{dr} = 0$ where C is the intersection of $x^2+y^2 = 1$ with $z = 1$.

Solution:
This follows from the fundamental theorem of line integrals.

13) | T | F | If the flux of vector field is zero through any surface S in space, then the divergence of the field is zero everywhere in space.

Solution:
This follows from the divergence theorem.

14) | T | F | The curl of a gradient field $\vec{F}(x,y,z) = \nabla f(x,y,z)$ is zero, if $f(x,y,z) = \sqrt{x^{10} + y^{10}z^2}$.

Solution:
Yes, we have seen this follows from Clairot.

15) | T | F | The line integral of the curl of a vector field $\vec{F}(x,y,z) = \langle x,y,z \rangle$ along a circle in the $xy-$ plane is zero.

Solution:
The flux integral of the curl through a closed surface is zero.

16) | T | F | For a solid E which is rotationally symmetric around the z-axes, the integral $\iiint_E \sqrt{x^2+y^2}\; dxdydz$ is equal to the volume of the solid.

Solution:
It is already different for the sphere.

17) | T | F | The curvature of the curve $\vec{r}(t) = \langle 1+2\cos(1+t), 1+2\sin(1+t) \rangle$ is constant equal to $1/2$ everywhere.

Solution:
This is a circle of radius 2. It has curvature 1/2.

18) | T | F | The directional derivative of $f(x,y,z) = \text{div}(\vec{F}(x,y,z))$ of the divergence of the vector field $\vec{F} = \langle P,Q,R \rangle$ in the direction $\vec{v} = \langle 1,0,0 \rangle$ is $P_{xx}+Q_{xy}+R_{xz}$.

Solution:
Yes by definition $\text{div}(\vec{F}(x,y,z)) = P_x(x,y,z) + Q_y(x,y,z) + R_z(x,y,z)$. The directional derivative in the $\langle 1,0,0\rangle$ direction is the partial derivative.

19) | T | **F** | $\int_0^{2\pi} \int_0^{2\pi} r \, d\theta \, dr = \int_0^{2\pi} \int_0^{2\pi} 1 \, dxdy$.

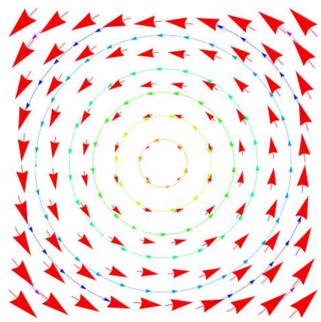
Solution:
Wrong bound. The right integral integrates over a disc.

20) | **T** | F | The set $\{\phi = \pi, \rho > 0\}$ in spherical coordinates is the negative z-axis.

Solution:
If $\phi = \pi$, then we are on the negative z-axes.

Problem 2) (10 points) No justifications are necessary.

a) (4 points) Match the vector fields with the definitions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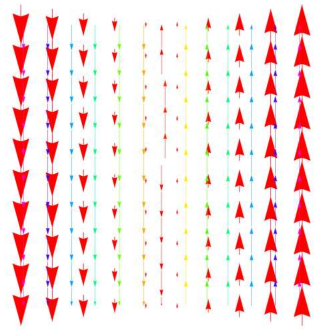
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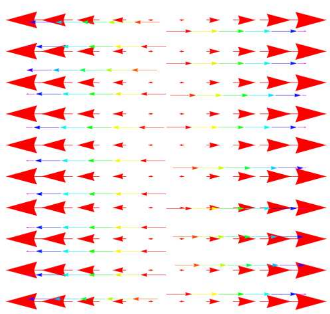
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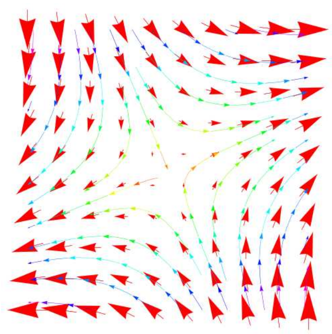
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Enter 1-4	vector field
	$\vec{F}(x,y) = \langle x+y, x-y \rangle$
	$\vec{F}(x,y) = \langle 0, x \rangle$
	$\vec{F}(x,y) = \langle -y, x \rangle$
	$\vec{F}(x,y) = \langle x, 0 \rangle$

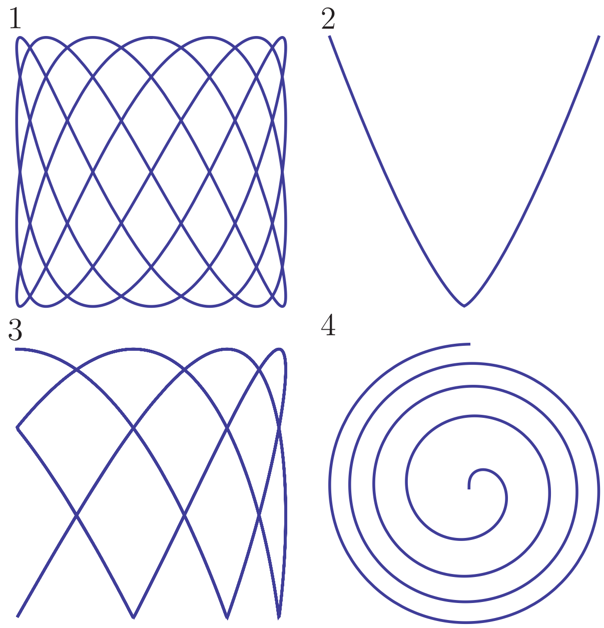
Solution:

b) (3 points) Match the partial differential equations (PDE's) with their names

1) Wave equation
2) Heat equation
3) Transport equation
4) Burgers equation

Enter 1-4	PDE
	$u_t - u_{xx} = 0$
	$u_{tt} - u_{xx} = 0$
	$u_t - u_x = 0$

c) (3 points) Match the curves



Enter 1-4	Parametrized curve
	$\vec{r}(t) = \langle \cos(4t), \sin(7t) \rangle$
	$\vec{r}(t) = \langle \sqrt{t}\sin(t), \sqrt{t}\cos(t) \rangle$
	$\vec{r}(t) = \langle \lvert\cos(4t)\rvert, \lvert\sin(7t)\rvert \rangle$
	$\vec{r}(t) = \langle t^3, t^4 \rangle$

Solution:
a) 4,2,1,3
b) 2,1,3
c) 1,4,3,2

Problem 3) (10 points) No justifications are necessary

a) (6 points) Check the boxes which apply. Leave the other boxes empty. The expression "involves XYZ" means that the formulation of the statement contains the object XYZ somewhere.

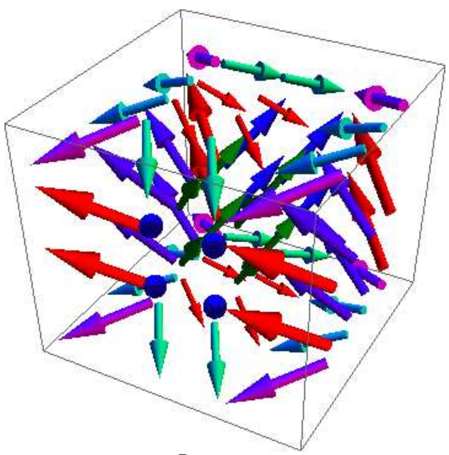
Statement	involves a curve	involves a surface	involves a vector field
Stokes theorem			
Divergence theorem			
Lagrange equations			
Fund. theorem line integrals			
Surface area formula			
Curvature formula			

Solution:

Statement	involves a curve	involves a surface	involves a vector field
Stokes theorem	yes	yes	yes
Divergence theorem		yes	yes
Lagrange equations	(yes)	(yes)	(yes)
Fund. theorem line integrals	yes		yes
Surface area formula		yes	
Curvature formula	yes		

With Lagrange, all answers were accepted because in two dimensions, a constraint could be considered a curve and in three dimension, the constraint could be considered a surface and the Lagrange equations could be a stretched to be an identity for gradient vector field. Even so, as the question was posed, there is no vector field mentioned, nor any curve or surface mentioned in the formulation of the result.

b) (4 points) Match the objects with their definitions



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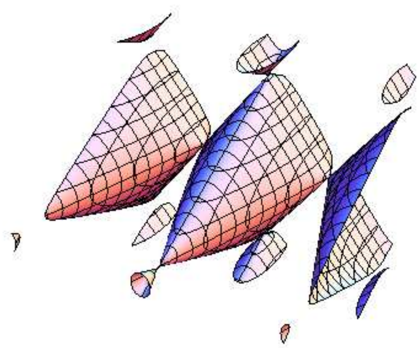

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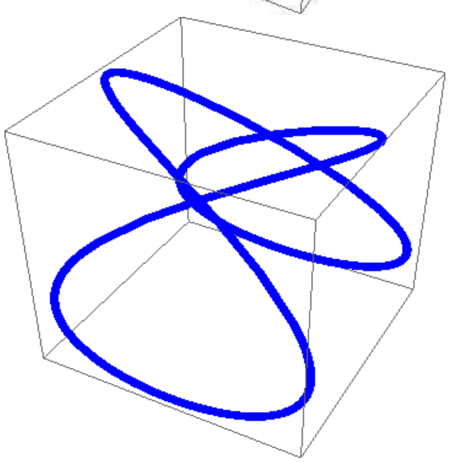

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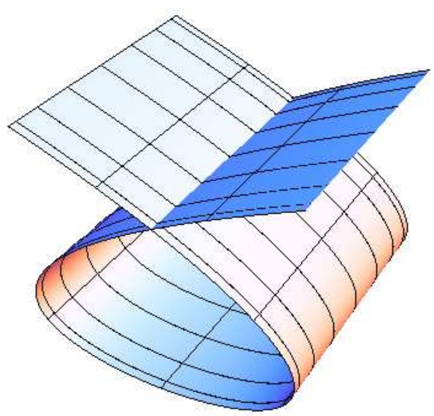
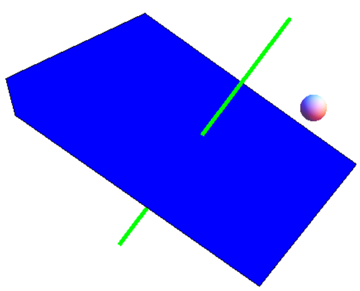


4

Enter 1-4	object definition
	$\vec{r}(t) = \langle \cos(3t), \sin(t), \cos(2t) \rangle$
	$\cos(3x) + \sin(y) + \cos(2z) = 1$
	$\vec{r}(t,s) = \langle \cos(3t), \sin(s), \cos(2t) \rangle$
	$\vec{F}(x,y,z) = \langle \cos(3x), \sin(y), \cos(2z) \rangle$

Solution:
b) 3,2,4,1.
The main difficulty was to identify which was the parametrized surface and which was the implicit surface. For the parametrized surface, the grid curves were your friend. If the variable "t" is kept constant, you see lines. Lots of them. The surface must be of cylindrical shape.

Problem 4) (10 points)

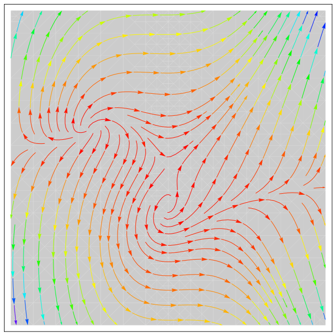


a) (5 Points) Write down a parametrization $\vec{r}(t)$ of the line which is perpendicular to the plane $x + 2y + z = 0$ and which passes through the origin.

b) (5 points) Find the distance of this line to the point $(3, 4, 5)$.

Solution:
a) $\langle t, 2t, t \rangle$
b) $|\langle 1,2,1 \rangle \times \langle 3,4,5 \rangle| / |\langle 1,2,1 \rangle| = |\langle 6,-2,-2 \rangle| / \langle [1,2,1 \rangle| = \sqrt{44}/\sqrt{6}$. The answer is $\boxed{\sqrt{22/3}}$ One student wrote the result as $\sqrt{66}/3$, the line being "route 66".

Problem 5) (10 points)



Find the place where the curl

$$f(x,y) = \text{curl}(\vec{F})(x,y) = Q_x(x,y) - P_y(x,y)$$

of the vector field

$$\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle = \langle x + y^2 + y, x^2y + 2x + y^2\rangle$$

is maximal under the constraint that the divergence

$$g(x,y) = \text{div}(\vec{F})(x,y) = P_x(x,y) + Q_y(x,y) \ .$$

is equal to 1. Find the functions f, g and solve the problem using Lagrange.

Solution:
We have

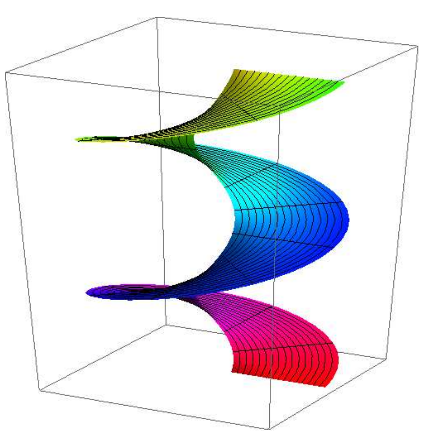
$$f(x,y) = 2xy + 2 - 2y - 1$$
$$g(x,y) = 1 + x^2 + 2y$$

The Lagrange equations are

$$\begin{aligned} 2y &= \lambda 2x \\ 2x - 2 &= 2\lambda \\ 1 + x^2 + 2y &= 1 \end{aligned}$$

Getting rid of λ gives $y = x(x-1)$ and plugging this into the constraint gave $3x^2 - 2x = 0$ which has the solution $x = 0$ or $x = 2/3$.
The solutions are $(0,0)$ and $(2/3, -2/9)$. At the first point the curl is 1 at the second $(2/3, -2/9)$ it is $158/81 > 1$. The second point $(2/3, -2/9)$ is the maximum.

Problem 6) (10 points)



a) (5 points) Find the surface area of the surface

$$\vec{r}(s,t) = \langle s\cos(t), s\sin(t), t\rangle$$

with $1 \leq s \leq 2, 0 \leq t \leq 4\pi$.

b) (5 points) Find the arc length of the curve

$$\vec{r}(t) = \langle \cos(t), \sin(t), t\rangle$$

with $0 \leq t \leq 4\pi$.

Hint. You can use without derivation the during lecture derived integral $\int \sqrt{x^2+1}\, dx = (x\sqrt{x^2+1} + \text{arcsinh}(x))/2$ and you can leave terms like arcsinh(2).

Solution:
a) The vector $\vec{r}_s \times \vec{r}_t = \langle \cos(t), \sin(t), 1 \rangle \times \langle -s\sin(t), s\cos(t) \rangle$ has length $\sqrt{1+s^2}$. The surface area is

$$\int_0^{4\pi} \int_1^2 \sqrt{1+s^2} \; ds \; dt \; .$$

which is

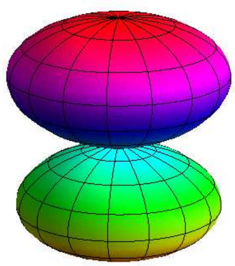
$$2\pi \int_1^2 \sqrt{1+s^2} \; ds = 2\pi(2\sqrt{5} - \sqrt{2} + \text{arcsinh}(2) - \text{arcsinh}(1)) \; .$$

b) Since the speed is constant $|\vec{r}\,'(t) = \sqrt{\cos^2(t) + \sin^2(t) + 1} = \sqrt{2}$, the arc length is

$$\int_0^{4\pi} |\vec{r}\,'(t)| \; dt = \int_0^{4\pi} \sqrt{2} \; dt$$

which is $\boxed{4\pi\sqrt{2}}$.

Problem 7) (10 points)



Find the volume of the solid given in spherical coordinates as

$$\rho(\phi, \theta) \leq \cos^2(\phi) \; .$$

with $0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi$.

Solution:

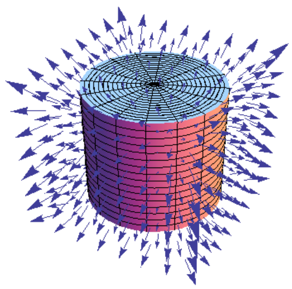
$$\int_0^{2\pi} \int_0^{\pi} \int_0^{\cos^2(\phi)} \rho^2 \sin(\phi) \; d\rho d\phi d\theta = (2/3)\pi \int_0^{\pi} \cos^6(\phi) \sin(\phi) \; d\phi = 4\pi/21 \; .$$

where substitution has given us

$$\int \cos^6(\phi) \sin(\phi) \; d\phi = -\cos^7(\phi)/7 + C \; .$$

The answer is $\boxed{4\pi/21}$.

Problem 8) (10 points)



Find the flux $\iint_S \vec{F} \cdot d\vec{S}$ of the vector field

$$\vec{F}(x, y, z) = \langle x^3, y^3, z + (1 - x^2 - y^2)(1 - z^2) \rangle$$

through the boundary S of the solid cylinder

$$E : x^2 + y^2 \leq 1,\ z^2 \leq 1 \ .$$

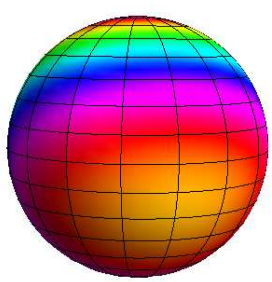
The boundary of S of the solid E is oriented outwards as usual.

Solution:
The vector field is a sum of $\langle x^3, y^3, z \rangle$ and $\langle 0, 0, (1 - x^2 - y^2)(1 - z^2) \rangle$. The flux of the later through the cylinder is zero because the field is zero on the cylinder surface. Therefore, we only need to compute the flux of $\langle x^3, y^3, z \rangle$ through the surface. We use the divergence theorem for that. The divergence is $3x^2 + 3y^2 + 1 = 1 + 3r^2$. To integrate this over the cylinder, best use cylindrical coordinates:

$$\int_0^1 \int_0^{2\pi} \int_{-1}^1 (1 + 3r^2) r \ dz d\theta dr = 2 \cdot 2\pi \cdot (5/4) = 5\pi \ .$$

The final answer is $\boxed{5\pi}$. It was possible to take the entire vector field and compute the triple integral but it was a bit more work.

Problem 9) (10 points)



Where on the sphere is the function

$$f(\phi, \theta) = \sin(\phi) + \sin(\theta)$$

extremal? Find all maxima, minima and saddle points of f as well as the global maxima and minima.

Remark. The variables ϕ, θ are the usual spherical coordinates variables. You are welcome of course to write $f(x, y) = \sin(x) + \sin(y)$ and look for solutions $0 \leq x \leq \pi, 0 \leq y < 2\pi$.

Solution:
To find the critical points, we find the gradient

$$\nabla f(x,y) = \langle \cos(x), \cos(y) \rangle \ .$$

It is zero for $x = \pi/2$ and $y = \pi/2, 3\pi/2$. There are two critical points.

$(\pi/2, \pi/2)$	$D > 0$	$f_{xx} < 0$	maximum
$(\pi/2, 3\pi/2)$	$D < 0$		saddle point

We have used that the discriminant is $D(x,y) = \sin(x)\sin(y)$ and that $f_{xx} = -\sin(x)$. There is a local max at $(\pi/2, \pi/2)$ and a saddle point at $\pi/2, 3\pi/2$. Additionally, we have to look what happens at the boundary $\phi = 0$ and $\phi = \pi$. For $\theta = 3\pi/2, \phi = 0$ we have the value -1 which is a global minimum. This is a solution if the problem is considered on the parameter domain $0 \leq x \leq \pi, 0 \leq y < 2\pi$ as indicated in the remark. The function $f(\phi, \theta)$ however is not continuous at the poles when considered a function on the sphere. The function value gets arbitrarily close to -1 but there is no global minimum away from the poles. *[There were only 2 from 10 points assigned to the global extrema part. Both answers (1. no global minimum exists or 2. there is a global minimum taking value -1 was given full credit in this problem.]*

Problem 10) (10 points)

The "sin-log" function $\sin(x)/\log(x)$ has no known antiderivative. Determined to overcome this obstacle, we nevertheless integrate

$$\int_0^1 \int_{e^y}^e \frac{\sin(x)}{\log(x)} \, dxdy \ .$$

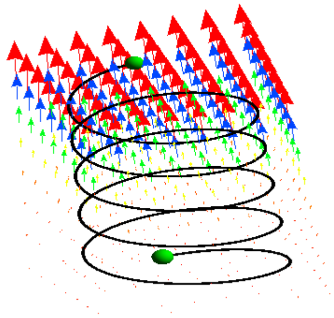
Note that $\log(x)$ denotes the natural logarithm. The ln notation is for greenhorns.

Solution:
Change the order of integration solves the riddle. As usual, it was crucial to make a picture and to label the bounds.

$$\int_1^e \int_0^{\log(x)} \frac{\sin(x)}{\log(x)} \, dydx$$

$\cos(1) - \cos(e)$

Problem 11) (10 points)



Find the line integral

$$\int_0^{10\pi} \vec{F} \cdot \, \vec{dr} \ ,$$

where $\vec{F}(x,y,z) = \langle x^2+1, y^2, z^3+x^2 \rangle$ and where $\vec{r}(t)$ is the spiral $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$.

Solution:
Split up the vector field into a gradient field $\vec{F}(x,y,z) = \langle x^2+1, y^2, z^3 \rangle$ and a part $\langle 0,0,x^2 \rangle$. The line integral of the first field can be computed with the fundamental theorem of line integrals because

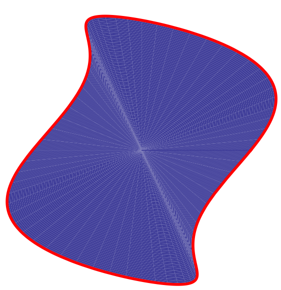
$$f(x,y,z) = x^3/3 + x + y^3/3 + z^4/4$$

is a potential function. The line integral of $\langle x^2+1, y^2, z^3 \rangle$ is the difference between the potential values at $(1,0,10\pi)$ and $(1,0,0)$ which is $(10\pi^4)/4$. The second line integral with the field $\langle 0,0,x^2 \rangle$ can be computed directly

$$\int_0^{10\pi} \langle 0,0,\cos^2(t) \rangle \cdot \langle 0,0,1 \rangle \, dt = 5\pi \ .$$

The total answer is $\boxed{(10\pi)^4/4 + 5\pi}$. Some have attempted to solve the problem using Stokes theorem. This is possible but one has to be careful to find a suitable surface which has the spiral as a boundary and also to add more curve to get all the boundary.

Problem 12) (10 points)



Find the area of the region enclosed by the curve

$$\vec{r}(t) = \langle \cos(t) + \frac{\sin(3t)}{3}, 3\sin(t) \rangle$$

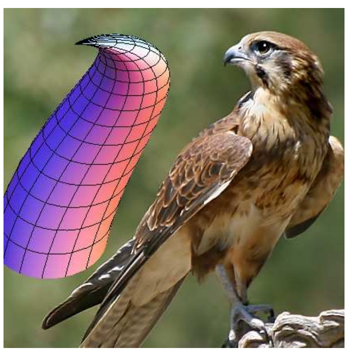
where $0 \leq t \leq 2\pi$.

Solution:
This is a problem for Greens theorem. Take a vector field like $\langle -y, 0\rangle$ with curl 1 and compute the line integral along the curve

$$\int_0^{2\pi} \langle -3\sin(t), 0\rangle \cdot \langle -\sin(t) + \cos(3t), 3\cos(t)\rangle = 3\pi \ .$$

The answer is $\boxed{3\pi}$. The integral $\int \sin(t)\cos(3t)\ dt$ could be solved by using integration by parts twice but it can also be seen to be zero by noting that the function is odd (the integrals from 0 to π and the part from π to 2π cancel).

Problem 13) (10 points)



Find the flux $\iint_S \mathrm{curl}(F)\ dS$ of the curl of the vector field

$$\vec{F}(x, y, z) = \langle -2y, 2x, 6e^{xy^2}\rangle$$

through the **falcon** surface

$$\vec{r}(s, t) = \langle \cos(t)\sin(s) + \frac{\sin(3s)}{2}, \sin(t)\sin(s) + \sin(2s), 4\cos(s)\rangle$$

parametrized by $0 \le t < 2\pi$ and $0 \le s \le \pi/2$.

Solution:
Stokes theorem relates the flux with the line integral of $\vec{F}$ along the boundary. The boundary part $s = 0$ is only a point (the beak) and does not contribute. The part $s = \pi/2$ is the boundary. It is a circle

$$\vec{r}(t) = \langle \cos(t) - 1/2, \sin(t), 0\rangle \ .$$

The vector field is $\vec{F}(\vec{r}(t)) = \langle -2\sin(t), 2\cos(t), 6e^{\cos(t)\sin^2(t)}\rangle$. The line integral is $\boxed{4\pi}$