1: Geometry and Distance

Multivariable calculus takes place is the two-dimensional **plane** and the three dimensional **space**.

A point in the **plane** has two **coordinates** $P = (x, y)$. A point in space is determined by three coordinates $P = (x, y, z)$. The signs of the coordinates define 4 **quadrants** in the plane and 8 **octants** in space. These regions intersect at the **origin** $O = (0, 0)$ or $O = (0, 0, 0)$ and are bound by **coordinate axes** $\{y = 0\}$ and $\{x = 0\}$ or **coordinate planes** $\{x = 0\}, \{y = 0\}, \{z = 0\}$.

In two dimensions, the x-coordinate usually directs to the "east" and the y-coordinate points "north". In three dimensions, the usual coordinate system has the xy-plane as the "ground" and the z-coordinate axes pointing "up".

1 $P = (2, -3)$ is in the forth quadrant of the plane and $P = (1, 2, 3)$ is in the positive octant of space. The point $(0, 0, -5)$ is on the negative z axis. The point $(1, 2, -3)$ is below the xy-plane.

2 **Problem.** Find the midpoint M of $P = (1, 2, 5)$ and $Q = (-3, 4, 7)$. **Answer.** The midpoint is obtained by taking the average of each coordinate $M = (P + Q)/2 = (-1, 3, 6)$.

3 In computer graphics of photography, the xy-plane contains the retina or film plate. The z coordinate measures the distance towards the viewer. In this **photographic coordinate** system, your eyes and chin define are points in the plane $z = 0$ and the nose points in the z direction. If the midpoint of your eyes is the origin of the coordinate system and your eyes have the coordinates $(1, 0, 0)$, $(-1, 0, 0)$, then the tip of your nose might have the coordinates $(0, -1, 1)$.

The **Euclidean distance** between two points $P = (x, y, z)$ and $Q = (a, b, c)$ in space is defined as $d(P, Q) = \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}$.

Note that this is a **definition** not a result. It is only motivated by **Pythagoras theorem**. We will **prove** the later.

4 **Problem:** Find the distance $d(P, Q)$ between the points $P = (1, 2, 5)$ and $Q = (-3, 4, 7)$ and verify that $d(P, M) + d(Q, M) = d(P, Q)$. **Answer:** The distance is $d(P, Q) = \sqrt{4^2 + 2^2 + 2^2} = \sqrt{24}$. The distance $d(P, M)$ is $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. The distance $d(Q, M)$ is $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. Indeed $d(P, M) + d(M, Q) = d(P, Q)$.

Remarks.
1) Distances can be introduced more abstractly: take any nonnegative function $d(P, Q)$ which satisfies the **triangle inequality** $d(P, Q) + d(Q, R) \geq d(P, R)$ and $d(P, Q) = 0$ if and only if $P = Q$. A set X with such a distance function d is called a **metric space**. Examples of distances are the **Manhatten distance** $d_m(P, Q) = |x - a| + |y - b|$, the **quartic distance** $d_4(P, Q) = ((x - a)^4 + (y - b)^4$ or the **Fermat distance** $d_f(x, y) = d(x, y)$ if $y > 0$ and $d_f(x, y) = 1.33 d(x, y)$ if $y < 0$. The constant 1.33 is the **refractive index** and models the

upper half plane being filled with air and the lower half plane with water. Shortest paths are bent at the water surface. Each of these distances d, d_m, d_4, d_f make the plane a different metric space.
2) It is **symmetry** which distinguishes the Euclidean distance as the most natural one. The Euclidean distance is determined by $d((1,0,0),(0,0,0)) = 1$, rotational and translational and scale symmetry $d(\lambda P, \lambda Q) = \lambda d(P,Q)$.
3) We usually work with a **right handed coordinate system**, where the x, y, z axes can be matched with the thumb, pointing and middle finger of the **right hand**. The photographers coordinate system is an example of a **left handed coordinate system**. The x, y, z axes are matched with the thumb and pointing finger and middle finger of the left hand. Nature is not oblivious to parity. Some laws of particle physics are different when they are observed in a mirror. Coordinate systems with different parity can not be rotated into each other.
4) When dealing with problems in the plane, we leave the z coordinate away and have $d(P,Q) = \sqrt{(x-a)^2 + (y-b)^2}$, where $P = (x,y), Q = (a,b)$.

Points, **curves**, **surfaces** and **solids** are geometric objects which can be described with **functions of several variables**. An example of a curve is a line, an example of a surface is a plane, an example of a solid is the interior of a sphere. We focus in this first lecture on spheres or circles.

A **circle** of radius r centered at $P = (a,b)$ is the collection of points in the plane which have distance r from P.

A **sphere** of radius ρ centered at $P = (a,b,c)$ is the collection of points in space which have distance ρ from P. The equation of a sphere is $(x-a)^2+(y-b)^2+(z-c)^2 = \rho^2$.

An **ellipse** is the collection of points P in the plane for which the sum $d(P,A) + d(P,B)$ of the distances to two points A, B is a fixed constant l larger than $d(A,B)$. This allows to draw the ellipse with a string of length l attached at A, B. An algebraic equivalent description is the set of points satisfying an equation $x^2/a^2 + y^2/b^2 = 1$.

5 **Problem:** Is the point $(3,4,5)$ outside or inside the sphere $(x-2)^2+(y-6)^2+(z-2)^2 = 16$?
Answer: The distance of the point to the center of the sphere is $\sqrt{1+4+9}$ which is smaller than 4 the radius of the sphere. The point is inside.

6 **Problem:** Find an algebraic expression for the set of all points for which the sum of the distances to $A = (1,0)$ and $B = (-1,0)$ is equal to 3. **Answer:** Square the equation $\sqrt{(x-1)^2+y^2} + \sqrt{(x+1)^2+y^2} = 3$, separate the remaining single square root on one side and square again. Simplification gives $20x^2 + 36y^2 = 45$ which is equivalent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a, b can be computed as follows: because $P = (a,0)$ satisfies this equation, $d(P,A) + d(P,B) = (a-1) + (a+1) = 3$ so that $a = 3/2$. Similarly, the point $Q = (0,b)$ satisfying it gives $d(Q,A) + d(P,B) = 2\sqrt{b^2+1} = 3$ or $b = \sqrt{5}/2$.

Here is a verification with the computer algebra system Mathematica. Writing $L = d(P,A)$ and $M = d(P,B)$ we simplify the equation $L^2 + M^2 = 3^2$. The part without square root is $((L+M)^2 + (L-M)^2)/2 - 3^2$. The remaining square root is $((L+M)^2 - (L-M)^2)/2$. Now square both and set them equal to see the equation $20x^2 + 36y^2 = 45$.

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L=Sqrt[(x-1)^2+y^2]; M=Sqrt[(x+1)^2+y^2];
Simplify[(((L+M)^2 + (L-M)^2)/2-3^2)^2 == (((L+M)^2-(L-M)^2)/2)^2]
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The **completion of the square** of an equation $x^2 + bx + c = 0$ is the idea to add $(b/2)^2 - c$ on both sides to get $(x + b/2)^2 = (b/2)^2 - c$. Solving for x gives the solution $x = -b/2 \pm \sqrt{(b/2)^2 - c}$.

7 The equation $2x^2 - 10x + 12 = 0$ is equivalent to $x^2 + 5x = -6$. Adding $(5/2)^2$ on both sides gives $(x + 5/2)^2 = 1/4$ so that $x = 2$ or $x = 3$.

8 The equation $x^2 + 5x + y^2 - 2y + z^2 = -1$ is after completion of the square $(x + 5/2)^2 - 25/4 + (y - 1)^2 - 1 + z^2 = -1$ or $(x - 5/2)^2 + (y - 1)^2 + z^2 = (5/2)^2$. We see a sphere **center** $(5/2, 1, 0)$ and **radius** $5/2$.

The method is due to **Al-Khwarizmi** who lived from 780-850 and used it as a method to solve quadratic equations. Even so Al-Khwarizmi worked with numerical examples, it is one of the first important steps of algebra. His work *"Compendium on Calculation by Completion and Reduction"* was dedicated to the Caliph **al Ma'mun**, who had established research center called "House of Wisdom" in Baghdad. [1] In an appendix to "Geometry" of his "Discours de la méthode" which appeared in 1637, **René Descartes** promoted the idea to use algebra to solve geometric problems. Even so Descartes mostly dealt with ruler-and compass constructions, the rectangular coordinate system is now called the **Cartesian coordinate system**. His ideas profoundly changed mathematics. Ideas do not grow in a vacuum. Davis and Hersh write that in its current form, Cartesian geometry is due as much to Descartes own contemporaries and successors as to himself. [2]

What happens in higher dimensions? A point in four dimensional space for example is labeled with four coordinates (t, x, y, z). How many hyper chambers is space are obtained when using coordinate hyperplanes $t = 0, x = 0, y = 0, z = 0$ as walls? Answer: There are 16 hyper-regions and each of them contains one of the 16 points (x, y, z, w), where x, y, z, w are either $+1$ or -1.

Homework

1 Describe and sketch the set of points $P = (x, y, z)$ in three dimensional space $\mathbf{R}^3$ represented by

[1] The book "The mathematics of Egypt, Mesopotamia,China, India and Islam, a Sourcebook, Ed Victor Katz, page 542 contains translations of some of this work.

[2] An entertaining read is "Descartes Secret Notebook" by Amir Aczel.

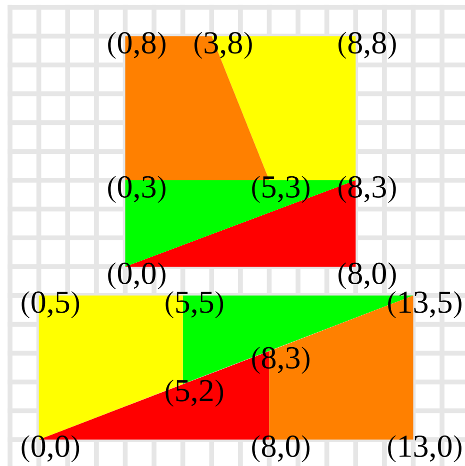
a) $(x-1)^2 + (z-1)^2 = 25$ c) $x^2y^2z^2 = 0$

b) $x - y - 2z = 14$ d) $y^2 = z$

2 a) Find the distances of $P = (7, 24, 0)$ to each of the 3 coordinate axes.

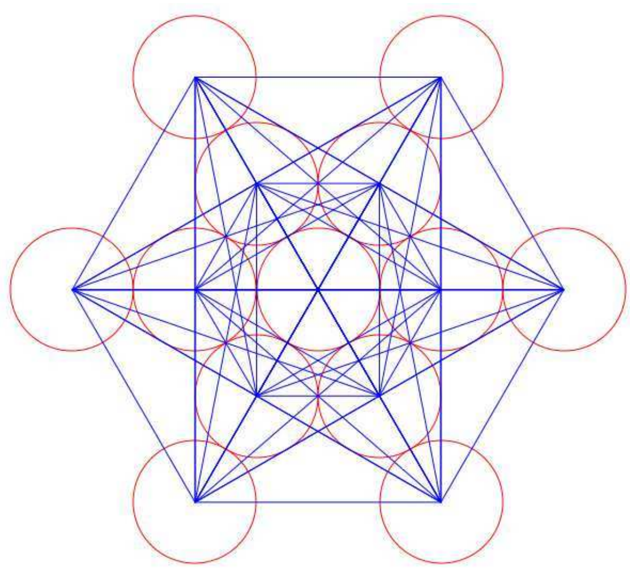
b) Find the distances of $P = (2, 3, 5)$ to each of the 3 coordinate planes.

3 The figure shows two rectangles. One has the area $8 \cdot 8 = 64$. The other has the area $65 = 13 \cdot 5$. But these triangles are made up by triangles or trapezoids which match. Measure various distances to see what is going on.



4 Find the center and radius of the sphere $x^2 + 2x + y^2 - 16y + z^2 + 10z + 54 = 0$. Describe the traces of this surface, its intersection with each of the coordinate planes.

5 Metatron's cube is a geometric figure which appears in "sacret geometry". It serves also as an inspiration for artists and been labeled with "magical" properties because it contains images of polyhedra. Make list of distances which appear between centers of circles.



2: Vectors and Dot Product

Two points $P = (a, b, c)$ and $Q = (x, y, z)$ in space define a **vector** $\vec{v} = \langle x - a, y - b - z - c\rangle$. It connecdts P with Q and we write also $\vec{v} = \vec{PQ}$. The real numbers numbers p, q, r in a vector $\vec{v} = \langle p, q, r\rangle$ are called the **components** of $\vec{v}$.

Vectors can be drawn **everywhere** in space but two vectors with the same components are considered **equal**. Vectors can be translated into each other if and only if their components are the same. If a vector $\vec{v}$ starts at the origin $O = (0, 0, 0)$, then $\vec{v} = \langle p, q, r\rangle$ heads to the point (p, q, r). One can therefore identify points $P = (a, b, c)$ with vectors $\vec{v} = \langle a, b, c\rangle$ attached to the origin. For clarity, we often draw an arrow on top of vectors and if $\vec{v} = \vec{PQ}$ then P is the "tail" and Q is the "head" of the vector. To distinguish vectors from points, it is custom to different brackets and write $\langle 2, 3, 4\rangle$ for vectors and $(2, 3, 4)$ for points.

The **sum** of two vectors is $\vec{u} + \vec{v} = \langle u_1, u_2\rangle + \langle v_1, v_2\rangle = \langle u_1 + v_1, u_2 + v_2\rangle$. The **scalar multiple** $\lambda\vec{u} = \lambda\langle u_1, u_2\rangle = \langle\lambda u_1, \lambda u_2\rangle$. The difference $\vec{u} - \vec{v}$ can best be seen as the addition of $\vec{u}$ and $(-1) \cdot \vec{v}$.

The vectors $\vec{i} = \langle 1, 0\rangle$, $\vec{j} = \langle 0, 1\rangle$ are called **standard basis vectors** in the plane. In space, one has the basis vectors $\vec{i} = \langle 1, 0, 0\rangle$, $\vec{j} = \langle 0, 1, 0\rangle$, $\vec{k} = \langle 0, 0, 1\rangle$.

Every vector $\vec{v} = \langle p, q\rangle$ in the plane can be written as a combination $\vec{v} = p\vec{i} + q\vec{j}$ of standard basis vectors and every vector $\vec{v} = \langle p, q, r\rangle$ in space can be written as $\vec{v} = p\vec{i} + q\vec{j} + r\vec{k}$. Vectors are abundant in applications. They appear in mechanics: if $\vec{r}(t) = \langle f(t), g(t)\rangle$ is a point in the plane which depends on time t, then $\vec{v} = \langle f'(t), g'(t)\rangle$ will be called the **velocity vector** at $\vec{r}(t)$. Here $f'(t), g'(t)$ are the derivatives. In physics, we often want to determine forces acting on objects. Forces are represented as vectors. In particular, electromagnetic or gravitational fields or velocity fields in fluids are described by vectors. Vectors appear also in computer science: the scalable vector graphics is a standard for the web for describing two-dimensional graphics. In quantum computation, rather than working with bits, one deals with **qbits**, which are vectors. Finally, **color** can be written as a vector $\vec{v} = \langle r, g, b\rangle$, where r is **red**, g is **green** and b is **blue** component of the color vector. An other coordinate system for color is $\vec{v} = \langle c, m, y\rangle = \langle 1 - r, 1 - g, 1 - b\rangle$, where c is **cyan**, m is **magenta** and y is **yellow**. Vectors appear in probability theory and statistics. On a finite probability space, a **random variable** is a vector.

The addition and scalar multiplication of vectors satisfy the laws you know from **arithmetic**. **commutativity** $\vec{u}+\vec{v} = \vec{v}+\vec{u}$, **associativity** $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$ and $r*(s*\vec{v}) = (r*s)*\vec{v}$ as well as **distributivity** $(r+s)\vec{v} = \vec{v}(r+s)$ and $r(\vec{v}+\vec{w}) = r\vec{v}+r\vec{w}$, where $*$ denotes multiplication with a scalar.

The **length** $|\vec{v}|$ of a vector $\vec{v} = \vec{PQ}$ is defined as the distance $d(P, Q)$ from P to Q. A vector of length 1 is called a **unit vector**. If $\vec{v} \neq \vec{0}$, then $\vec{v}/|\vec{v}|$ is a unit vector.

1 $|\langle 3, 4\rangle| = 5$ and $|\langle 3, 4, 12\rangle| = 13$. Examples of unit vectors are $|\vec{i}| = |\vec{j}| = \vec{k}| = 1$ and $\langle 3/5, 4/5\rangle$ and $\langle 3/13, 4/13, 12/13\rangle$. The only vector of length 0 is the zero vector $|\vec{0}| = 0$.

The **dot product** of two vectors $\vec{v} = \langle a, b, c \rangle$ and $\vec{w} = \langle p, q, r \rangle$ is defined as $\vec{v} \cdot \vec{w} = ap + bq + cr$.

Remarks.
a) Different notations for the dot product are used in different mathematical fields. while pure mathematicians write $\vec{v} \cdot \vec{w} = (\vec{v}, \vec{w})$, one can see $\langle \vec{v} | \vec{w} \rangle$ in quantum mechanics or $v_i w^i$ or more generally $g_{ij} v^i w^j$ in general relativity. The dot product is also called **scalar product** or **inner product**.
b) Any product $g(v, w)$ which is linear in v and w and satisfies the symmetry $g(v, w) = g(w, v)$ and $g(v, v) \geq 0$ and $g(v, v) = 0$ if and only if $v = 0$ can be used as a dot product. An example is $g(v, w) = 3v_1 w_1 + 2v_2 w_2 + v_3 w_3$.

The dot product determines distance and distance determines the dot product.

Proof: Lets write $v = \vec{v}$ in this proof. Using the dot product one can express the length of v as $|v| = \sqrt{v \cdot v}$. On the other hand, from $(v + w) \cdot (v + w) = v \cdot v + w \cdot w + 2(v \cdot w)$ can be solved for $v \cdot w$:

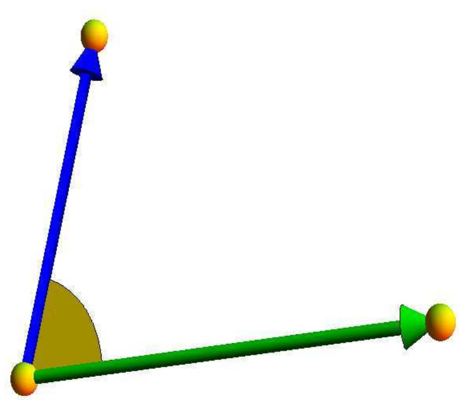
$$v \cdot w = (|v + w|^2 - |v|^2 - |w|^2)/2 \ .$$

The **Cauchy-Schwarz inequality** tells $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$.

Proof. We can assume $|w| = 1$ by rescaling the equation. Now plug in $a = v \cdot w$ into the equation $0 \leq (v - aw) \cdot (v - aw)$ to get $0 \leq (v - (v \cdot w)w) \cdot (v - (v \cdot w)w) = |v|^2 + (v \cdot w)^2 - 2(v \cdot w)^2 = |v|^2 - (v \cdot w)^2$ which means $(v \cdot w)^2 \leq |v|^2$.

Having established this, it is possible to give a clear definition of what an **angle** is without referring to geometric pictures:

The **angle** between two nonzero vectors is defined as the unique $\alpha \in [0, \pi]$ which satisfies $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$.



Al Kashi's theorem: If a, b, c are the side lengths of a triangle ABC and α is the angle opposite to c, then $a^2 + b^2 = c^2 - 2ab\cos(\alpha)$.

Proof. Define $\vec{v} = \vec{AB}, \vec{w} = \vec{AC}$. Because $c^2 = |\vec{v} - \vec{w}|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = |\vec{v}|^2 + |\vec{w}|^2 - 2\vec{v} \cdot \vec{w}$, We know $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$ so that $c^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}| \cdot |\vec{w}| \cos(\alpha) = a^2 + b^2 - 2ab\cos(\alpha)$.

The angle definition works in any space with a dot product. In **statistics** you work with vectors of n components. They are called **data** or random variables and $\cos(\alpha)$ is called the **correlation** between two random variables $\vec{v}, \vec{w}$ of zero **expectation** $\mathrm{E}[\vec{v}] = (v_1 + \cdots + v_n)/n$. The dot product $v_1 w_1 + \ldots + v_n w_n$ is then the **covariance**, the length $|v|$ is the **standard deviation** and denoted by $\sigma(v)$. The formula $\mathrm{Corr}[v, w] = \mathrm{Cov}[v, w]/(\sigma(v)\sigma(w))$ for the correlation is the familiar angle formula we have seen. It is geometry in n dimensions. We mention this only to convince you that the geometry we do here can be applied to much more. All the computations we have done go through verbatim.

The **triangle inequality** tells $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$.

Proof: $|\vec{u}+\vec{v}|^2 = (\vec{u}+\vec{v}) \cdot (\vec{u}+\vec{v}) = \vec{u}^2 + \vec{v}^2 + 2\vec{u} \cdot \vec{v} \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u} \cdot \vec{v}| \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u}| \cdot |\vec{v}| = (|\vec{u}| + |\vec{v}|)^2$.

Two vectors are called **orthogonal** or **perpendicular** if $\vec{v} \cdot \vec{w} = 0$. The zero vector $\vec{0}$ is orthogonal to any vector. For example, $\vec{v} = \langle 2, 3 \rangle$ is orthogonal to $\vec{w} = \langle -3, 2 \rangle$.

Having given precise definitions of all objects, we can now prove **Pythagoras theorem**:

Pythagoras theorem: if $\vec{v}$ and $\vec{w}$ are orthogonal, then $|v - w|^2 = |v|^2 + |w|^2$.

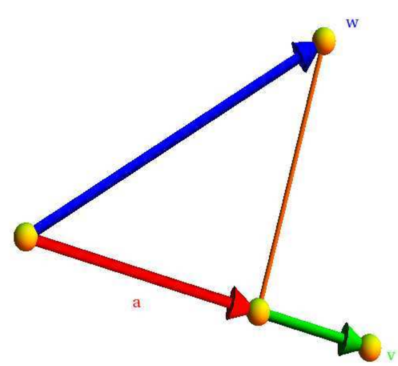
Proof: $(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} + 2\vec{v} \cdot \vec{w} = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w}$.

Remarks:
1) You have seen here something very powerful: results like Pythagoras (570-495BC) and Al Khashi (1380-1429) theorems were **derived from scratch** on a space V equipped with a dot product. The dot product appeared much later in mathematics (Hamilton 1843, Grassman 1844, Sylvester 1851, Cayley 1858). While we have used geometry as an intuition, the structure was built algebraically without any unjustified assumptions. This is mathematics: if we have a space V in which addition $\vec{v} + \vec{w}$ and scalar multiplication $\lambda\vec{v}$ is given and in which a dot product is defined, then all the just derived results apply. We have **not used** results of Al Khashi or Pythagoras but we have **derived** them and additionally obtained a **clear definition** what an angle is.
2) The derivation you have seen works in any dimension. Why do we care about higher dimensions? As already mentioned, a compelling motivation is **statistics**. Given 12 data points like the average monthly temperatures in a year, we deal with a 12 dimensional space. Geometry is useful to describe data. Pythagoras theorem is the property that the variance of two uncorrelated random variables adds up with the formula $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$.
3) A far reaching generalization of the geometry you have just seen is obtained if the dot product $g(v, w)$ is allowed to depend on the place, where the two vectors are attached. This produces **Riemannian geometry** and allows to work with spaces which are intrinsically **curved**. This mathematics is important in **general relativity** which describes gravity in a geometric way and which is one of the pillars of modern physics. But it appears in daily life too. On a hot summer day, if you look close at an object a hot asphalt street, the object can appear distorted or flickers. The dot product depends on the temperature of the air. Light rays no more move on straight lines but gets bent. In extreme cases, when the curvature of light rays is larger than the curvature of the earth, it leads to **Fata morgana** effects: one can see objects which are located beyond the

horizon.

4) Why don't we introduce vectors as algebraic objects $\langle 1,2,3 \rangle$? The reason is that in many applications like physics and even geometry, one wants to work with **affine vectors**, vectors which are attached at points. Forces for example act on points of a body, we will also look at vector fields, where at each point a vector is attached. Considering vectors with the same components as equal gives then the vector space in which we do the algebra. One could define a vector space axiomatically and then build from this affine vectors but it is a bit too abstract and not much is actually gained for the goals we have in mind. An even more modern point of view replaces affine vectors with members of a tangent bundle. But this is only necessary if one deals with spaces which are not flat. Even more general is to allow the space attached at each point to be a more general space like a "group" called fibres. So called "fibre bundles" are the framework of mathematical concepts which describe elementary particles or even space itself. Attaching a circle for example at each point leads to electromagnetism attaching classes of two dimensional matrices leads to the weak force and attaching certain three dimensional matrices leads to the strong force. Allowing this to happen in a curved framework incorporates gravity. One of the main challenges is to include quantum mechanics into that picture. Fundamental physics has become primarily the quest to answer the question "what is space"?

The vector $\mathrm{P}(\vec{v}) = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w}$ is called the **projection** of $\vec{v}$ onto $\vec{w}$. The **scalar projection** $\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|}$ is a signed length of the vector projection. Its absolute value is the length of the projection of $\vec{v}$ onto $\vec{w}$. The vector $\vec{b} = \vec{v} - P(\vec{v})$ is a vector orthogonal to the $\vec{w}$-direction.



2 For example, with $\vec{v} = \langle 0,-1,1 \rangle$, $\vec{w} = \langle 1,-1,0 \rangle$, $\mathrm{P}(\vec{v}) = \langle 1/2,-1/2,0 \rangle$. Its length is $1/\sqrt{2}$.

3 Projections are important in physics. For example, if you apply a wind force $\vec{F}$ to a car which drives in the direction $\vec{w}$ and P denotes the projection on $\vec{w}$ then $P(\vec{F})$ is the force which accelerates or slows down the car.

The projection allows to visualize the dot product. The absolute value of the dot product is the length of the projection. The dot product is positive if v points more towards to w, it is negative if v points away from it. In the next lecture we use the projection to compute distances between various objects.

Homework

1 Find a unit vector parallel to $\vec{u} - \vec{v} + \vec{w}$ if $\vec{u} = \langle 6, 7, 3 \rangle$ and $\vec{v} = \langle 2, 2, 3 \rangle$ and $\vec{w} = \langle -2, -1, 1 \rangle$.

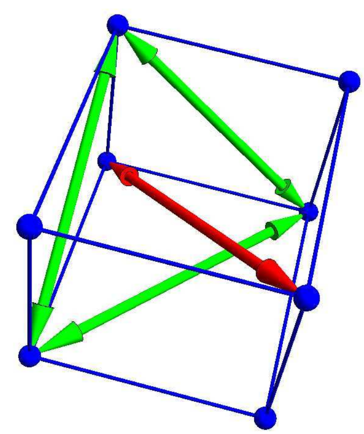
An **Euler brick** is a cuboid of dimensions a, b, c such that all face diagonals are integers.

a) Verify that $\vec{v} = \langle a, b, c \rangle = \langle 240, 117, 44 \rangle$ is a vector which leads to an Euler brick.

It had been found by Halcke in 1719.

2 b) (*) Verify that $\langle a, b, c \rangle = \langle u(4v^2 - w^2), v(4u^2 - w^2), 4uvw \rangle$ leads to an Euler brick if $u^2 + v^2 = w^2$.

(Sounderson 1740) If also the space diagonal $\sqrt{a^2 + b^2 + c^2}$ is an integer, an Euler brick is called **perfect**. Nobody has found one, nor proven that it can not exist.

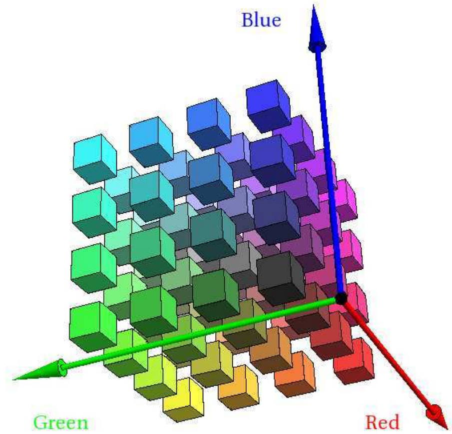


3 **Colors** are encoded by vectors $\vec{v} = \langle$ red, brightgreen, blue $\rangle$. The red, green and blue components of $\vec{v}$ are all real numbers in the interval $[0, 1]$.

a) Determine the angle between the colors yellow and cyan.

b) What is the projection of the mixture $(\vec{v} + \vec{w})/2$ of magenta and orange onto blue?

(0,0,0)	black	(0,0,1)	blue
(1,1,1)	white	(1,1,0)	yellow
$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	gray	(1,0,1)	magenta
(1,0,0)	red	(0,1,1)	cyan
(0,1,0)	green	$(1, \frac{1}{2}, 0)$	orange
$(0, 1, \frac{1}{2})$	vivid	$(1, 1, \frac{1}{2})$	khaki
$(1, \frac{1}{2}, \frac{1}{2})$	pink	$(\frac{1}{2}, \frac{1}{4}, 0)$	brown



4 Find the angle between the diagonal of the unit cube and one of the diagonal of one of its faces. Assume that the two diagonals go through the same edge of the cube. You can leave the answer in the form $\cos(\alpha) = ...$

5 Assume $\vec{v} = \langle -4, 2, 2\rangle$ and $\vec{w} = \langle 3, 0, 4\rangle$.
a) Find the vector projection of $\vec{v}$ onto $\vec{w}$.
b) Find the scalar component of $\vec{v}$ on $\vec{w}$.

3: Cross product

The **cross product** of two vectors $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$ in the plane is the scalar $v_1 w_2 - v_2 w_1$.

To remember this, write it as a determinant of a matrix which is an array of numbers: take the product of the diagonal entries and subtract the product of the side diagonal. $\begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix}$.

The **cross product** of two vectors $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ in space is defined as the vector

$$\vec{v} \times \vec{w} = \langle v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \rangle \ .$$

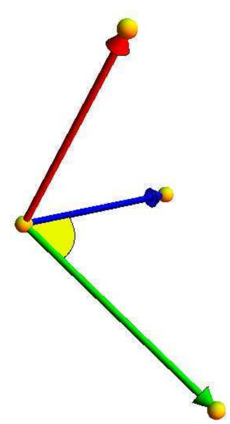
To remember it we write the product as a "determinant":

$$\begin{bmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \begin{bmatrix} i & & \\ & v_2 & v_3 \\ & w_2 & w_3 \end{bmatrix} - \begin{bmatrix} & j & \\ v_1 & & v_3 \\ w_1 & & w_3 \end{bmatrix} + \begin{bmatrix} & & k \\ v_1 & v_2 & \\ w_1 & w_2 & \end{bmatrix}$$

which is $\vec{i}(v_2 w_3 - v_3 w_2) - \vec{j}(v_1 w_3 - v_3 w_1) + \vec{k}(v_1 w_2 - v_2 w_1)$.

1 The cross product of $\langle 1, 2 \rangle$ and $\langle 4, 5 \rangle$ is $5 - 8 = -3$.

2 The cross product of $\langle 1, 2, 3 \rangle$ and $\langle 4, 5, 1 \rangle$ is $\langle -13, 11, -3 \rangle$.



The cross product $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$. The product is anti-commutative.

Proof. We verify for example that $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$ and look at the definition.

The **sin** formula: $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin(\alpha)$.

Proof: We verify first the **Lagrange's identity** $|\vec{v}\times\vec{w}|^2 = |\vec{v}|^2|\vec{w}|^2 - (\vec{v}\cdot\vec{w})^2$ by direct computation. Now, $|\vec{v} \cdot \vec{w}| = |\vec{v}||\vec{w}|\cos(\alpha)$.

The absolute value respectively length $|\vec{v} \times \vec{w}|$ defines the **area of the parallelogram** spanned by $\vec{v}$ and $\vec{w}$.

Note again that this was a definition of area so that nothing needs to be proven. But we want to make sure that the definition fits with our common intuition we have about area: $|\vec{w}|\sin(\alpha)$ is the height of the parallelogram with base length $|\vec{v}|$. We see from the sin-formula that the area does not change if we rotate the vectors around in space because both length and angle stay the same. Area also is linear in each of the vectors v and w. If we make v twice as long, then the area gets twice as large.

$\vec{v} \times \vec{w}$ is zero exactly if $\vec{v}$ and $\vec{w}$ are **parallel**, that is if $\vec{v} = \lambda\vec{w}$ for some real λ.

Proof. Use the sin formula and the fact that $\sin(\alpha) = 0$ if $\alpha = 0$ or $\alpha = \pi$.

The cross product can therefore be used to check whether two vectors are parallel or not. Note that v and $-v$ are also considered parallel even so sometimes one calls this anti-parallel.

The **trigonometric sin-formula**: if a, b, c are the side lengths of a triangle and α, β, γ are the angles opposite to a, b, c then $a/\sin(\alpha) = b/\sin(\beta) = c/\sin(\gamma$.

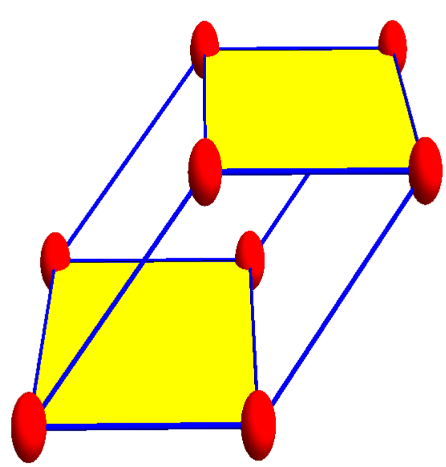
Proof. We express the area of the triangle in three different ways:

$$ab\sin(\gamma) = bc\sin(\alpha) = ac\sin(\beta) \ .$$

Divide the first equation by $\sin(\gamma)\sin(\alpha)$ to get one identity. Divide the second equation by $\sin(\alpha)\sin(\beta)$ to get the second identity.

3 If $\vec{v} = \langle a, 0, 0\rangle$ and $\vec{w} = \langle b\cos(\alpha), b\sin(\alpha), 0\rangle$, then $\vec{v} \times \vec{w} = \langle 0, 0, ab\sin(\alpha)\rangle$ which has length $|ab\sin(\alpha)|$.

The scalar $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ is called the **triple scalar product** of $\vec{u}, \vec{v}, \vec{w}$.



The absolute value of $[\vec{u}, \vec{v}, \vec{w}]$ defines the **volume of the parallelepiped** spanned by $\vec{u}, \vec{v}, \vec{w}$.

The **orientation** of three vectors is the sign of $[\vec{u}, \vec{v}, \vec{w}]$. It is positive if the three vectors form a right handed coordinate system.

Again, there was no need to prove anything because we have just **defined** volume and orientation. Let us still see why this fits with with our intuition about volume. The value $h = |\vec{u} \cdot \vec{n}|/|\vec{n}|$ is the height of the parallelepiped if $\vec{n} = (\vec{v} \times \vec{w})$ is a normal vector to the ground parallelogram of area $A = |\vec{n}| = |\vec{v} \times \vec{w}|$. The volume of the parallelepiped is $hA = (\vec{u} \cdot \vec{n}/|\vec{n}|)|\vec{v} \times \vec{w}|$ which simplifies to $\vec{u} \cdot \vec{n} = |(\vec{u} \cdot (\vec{v} \times \vec{w})|$ which is indeed the absolute value of the triple scalar product. The vectors $\vec{v}, \vec{w}$ and $\vec{v} \times \vec{w}$ form a **right handed coordinate system**. If the first vector $\vec{v}$ is your thumb, the second vector $\vec{w}$ is the pointing finger then $\vec{v} \times \vec{w}$ is the third middle finger of the right hand. For example, the vectors $\vec{i}, \vec{j}, \vec{i} \times \vec{j} = \vec{k}$ form a right handed coordinate system.

Since the triple scalar product is linear with respect to each vector we also see that volume is additive. Adding two equal parallelepipeds together for example gives a parallelepiped with a volume twice the volume.

4 **Problem:** Find the volume of a **cuboid** of width a length b and height c. **Answer**. The cuboid is a parallelepiped spanned by $\langle a, 0, 0\rangle$ $\langle 0, b, 0\rangle$ and $\langle 0, 0, c\rangle$. The triple scalar product is abc.

5 **Problem** Find the volume of the parallelepiped which has the vertices $O = (1, 1, 0), P = (2, 3, 1), Q = (4, 3, 1), R = (1, 4, 1)$. **Answer**: We first see that it is spanned by the vectors $\vec{u} = \langle 1, 2, 1\rangle$, $\vec{v} = \langle 3, 2, 1\rangle$, and $\vec{w} = \langle 0, 3, 1\rangle$. We get $\vec{v} \times \vec{w} = \langle -1, -3, 9\rangle$ and $\vec{u} \cdot (\vec{v} \times \vec{w}) = 2$. The volume is 2.

Homework

1 a) Find the volume of the parallelepiped for which the base parallelogram is given by the points $(1, 1, 1), (2, 0, 1), (0, 3, 1), (1, 2, 1)$ and which has an edge connecting $(1, 1, 1)$ with $(4, 5, 6)$.
b) Find the area of the base and use a) to get the height.

2 a) Assume $\vec{u} + \vec{v} + \vec{w} = \vec{0}$. Verify that $\vec{u} \times \vec{v} = \vec{v} \times \vec{w} = \vec{w} \times \vec{u}$.
b) Find $(\vec{u} + \vec{v}) \cdot (\vec{v} \times \vec{w})$ if $\vec{u}, \vec{v}, \vec{w}$ are unit vectors which are orthogonal to each other and $\vec{u} \times \vec{v} = \vec{w}$.

3 To find the equation $ax + by + cz = d$ for the plane which contains the point $P = (1, 2, 3)$ as well as the line which passes through $Q = (3, 4, 4)$ and $R = (1, 1, 2)$, we find a vector $\langle a, b, c\rangle$ normal to the plane and fix d so that P is in the plane.

4 (*) Verify the Lagrange formula

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$$

for general vectors $\vec{a}, \vec{b}, \vec{c}$ in space. The formula can be remembered as "BAC minus CAB".

5 Assume you know that the triple scalar product $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ between $\vec{u}, \vec{v}, \vec{w}$ is equal to 4. Find the values of $[\vec{v}, \vec{u}, \vec{w}]$ and $[\vec{u} + \vec{v}, \vec{v}, \vec{w}]$.

4: Lines and Planes

A point $P = (p, q, r)$ and a vector $\vec{v} = \langle a, b, c\rangle$ define the **line**

$$L = \{\langle p, q, r\rangle + t\langle a, b, c\rangle, t \in \mathbf{R}\ \} \ .$$

The line consists of all points obtained by adding a multiple of the vector $\vec{v}$ to the vector $\vec{OP} = \langle p, q, r\rangle$. The line contains the point P as well as a suitably translated copy of $\vec{v}$. Every vector contained in the line is necessarily parallel to $\vec{v}$. We think about the parameter t as "time". At time $t = 0$, we are at the point P, whereas at time $t = 1$ we are at $\vec{OP} + \vec{v}$.

If t is restricted to values in a **parameter interval** $[s, u]$, then $L = \{\langle p, q, r\rangle + t\langle a, b, c\rangle, s \leq t \leq u\ \}$ is a **line segment** which connects $\vec{r}(s)$ with $\vec{r}(u)$.

1 To get the line through $P = (1, 1, 2)$ and $Q = (2, 4, 6)$, form the vector $\vec{v} = \vec{PQ} = \langle 1, 3, 4\rangle$ and get $L = \{\langle x, y, z\rangle = \langle 1, 1, 2\rangle + t\langle 1, 3, 4\rangle;\}$. This can be written also as $\vec{r}(t) = \langle 1 + t, 1 + 3t, 2 + 4t\rangle$. If we write $\langle x, y, z\rangle = \langle 1, 1, 2\rangle + t\langle 1, 3, 4\rangle$ as a collection of equations $x = 1 + 2t, y = 1 + 3t, z = 2 + 4t$ and solve the first equation for t:

$$L = \{(x, y, z) \mid (x - 1)/2 = (y - 1)/3 = (z - 2)/4\ \} \ .$$

The line $\vec{r} = \vec{OP} + t\vec{v}$ defined by $P = (p, q, r)$ and vector $\vec{v} = \langle a, b, c\rangle$ with nonzero a, b, c satisfies the **symmetric equations**

$$\frac{x - p}{a} = \frac{y - q}{b} = \frac{z - r}{c} \ .$$

Proof. Each of these expressions is equal to t. These symmetric equations have to be modified a bit one or two of the numbers a, b, c are zero. If $a = 0$, replace the first equation with $x = p$, if $b = 0$ replace the second equation with $y = q$ and if $c = 0$ replace third equation with $z = r$.

2 Find the symmetric equations for the line through the two points $P = (0, 1, 1)$ and $Q = (2, 3, 4)$, we first form the parametric equations $\langle x, y, z\rangle = \langle 0, 1, 1\rangle + t\langle 2, 2, 3\rangle$ or $x = 2t, y = 1+2t, z = 1+3t$. Solving each equation for t gives the symmetric equation $x/2 = (y-1)/2 = (z - 1)/3$.

3 **Problem:** Find the symmetric equation for the z axes. **Answer:** This is a situation where $a = b = 0$ and $c = 1$. The symmetric equations are simply $x = 0, y = 0$. If two of the numbers a, b, c are zero, we have a coordinate plane. If one of the numbers are zero, then the line is contained in a coordinate plane.

A point P and two vectors $\vec{v}, \vec{w}$ define a **plane** $\Sigma = \{\vec{OP} + t\vec{v} + s\vec{w}$, where t, s are real numbers $\}$.

4 An example is $\Sigma = \{\langle x, y, z\rangle = \langle 1, 1, 2\rangle + t\langle 2, 4, 6\rangle + s\langle 1, 0, -1\rangle\ \}$. This is called the **parametric description** of a plane.

If a plane contains the two vectors $\vec{v}$ and $\vec{w}$, then the vector $\vec{n} = \vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$. Because also the vector $\vec{PQ} = \vec{OQ} - \vec{OP}$ is perpendicular to $\vec{n}$, we have $(Q - P) \cdot \vec{n} = 0$. With $Q = (x_0, y_0, z_0)$, $P = (x, y, z)$, and $\vec{n} = \langle a, b, c\rangle$, this means $ax + by + cz = ax_0 by_0 + cz_0 = d$. The plane is therefore described by a single equation $ax + by + cz = d$. We have just shown

The equation for a plane containing $\vec{v}$ and $\vec{w}$ and a point P is

$$ax + by + cz = d\ ,$$

where $\langle a, b, c\rangle = \vec{v} \times \vec{w}$ and d is obtained by plugging in P.

5 **Problem:** Find the equation of a plane which contains the three points $P = (-1, -1, 1)$, $Q = (0, 1, 1)$, $R = (1, 1, 3)$.
Answer: The plane contains the two vectors $\vec{v} = \langle 1, 2, 0\rangle$ and $\vec{w} = \langle 2, 2, 2\rangle$. We have $\vec{n} = \langle 4, -2, -2\rangle$ and the equation is $4x - 2y - 2z = d$. The constant d is obtained by plugging in the coordinates of a point to the left. In our case, it is $4x - 2y - 2z = -4$.

The **angle between the two planes** $ax + by + cz = d$ and $ex + fy + gz = h$ is defined as the angle between the two vectors $\vec{n} = \langle a, b, c\rangle$ and $\vec{m} = \langle e, f, g\rangle$.

6 Find the angle between the planes $x + y = -1$ and $x + y + z = 2$. Answer: find the angle between $\vec{n} = \langle 1, 1, 0\rangle$ and $\vec{m} = \langle 1, 1, 1\rangle$. It is $\arccos(2/\sqrt{6})$.

Finally, lets look at some distance formulas.

1) If P is a point and $\Sigma : \vec{n} \cdot \vec{x} = d$ is a plane containing a point Q, then

$$d(P, \Sigma) = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

is the distance between P and the plane. Proof: use the angle formula in the denominator. For example, to find the distance from $P = (7, 1, 4)$ to $\Sigma : 2x + 4y + 5z = 9$, we find first a a point $Q = (0, 1, 1)$ on the plane. Then compute

$$d(P, \Sigma) = \frac{|\langle -7, 0, -3\rangle \cdot \langle 2, 4, 5\rangle|}{|\langle 2, 4, 5\rangle|} = \frac{29}{\sqrt{45}}\ .$$

2) If P is a point in space and L is the line $\vec{r}(t) = Q + t\vec{u}$, then

$$d(P, L) = \frac{|(\vec{PQ}) \times \vec{u}|}{|\vec{u}|}$$

is the distance between P and the line L. Proof: the area divided by base length is height of parallelogram. For example, to compute the distance from $P = (2,3,1)$ to the line $\vec{r}(t) = (1,1,2) + t(5,0,1)$, compute

$$d(P,L) = \frac{|\langle -1,-2,1\rangle \times \langle 5,0,1\rangle|}{\langle 5,0,1\rangle} = \frac{|\langle -2,6,10\rangle|}{\sqrt{26}} = \frac{\sqrt{140}}{\sqrt{26}} \; .$$

3) If L is the line $\vec{r}(t) = Q + t\vec{u}$ and M is the line $\vec{s}(t) = P + t\vec{v}$, then

$$d(L,M) = \frac{|(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|}$$

is the distance between the two lines L and M. Proof: the distance is the length of the vector projection of $\vec{PQ}$ onto $\vec{u} \times \vec{v}$ which is normal to both lines. For example, to compute the distance between $\vec{r}(t) = (2,1,4) + t(-1,1,0)$ and M is the line $\vec{s}(t) = (-1,0,2) + t(5,1,2)$ form the cross product of $\langle -1,1,0\rangle$ and $\langle 5,1,2\rangle$ is $\langle 2,2,-6\rangle$. The distance between these two lines is

$$d(L,M) = \frac{|(3,1,2) \cdot (2,2,-6)|}{|\langle 2,2,-6\rangle|} = \frac{4}{\sqrt{44}} \; .$$

4) To get the distance between two planes $\vec{n} \cdot \vec{x} = d$ and $\vec{n} \cdot \vec{x} = e$, then their distance is

$$d(\Sigma,\Pi) = \frac{|e-d|}{|\vec{n}|}$$

Non-parallel planes have distance 0. Proof: use the distance formula between point and plane. For example, $5x + 4y + 3z = 8$ and $10x + 8y + 6z = 2$ have the distance

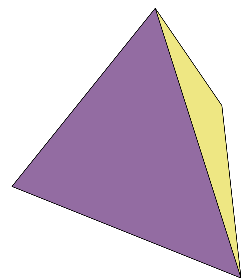
$$\frac{|8-1|}{|\langle 5,4,3\rangle|} = \frac{7}{\sqrt{50}} \; .$$

Here is a distance problem which has a great deal of application and motivates the material of the upcoming week: The **global positioning system** GPS uses the fact that a receiver can get the difference of distances to two satellites. Each GPS satellite sends periodically signals which are triggered by an atomic clock. While the distance to each satellite is not known, the difference from the distances to two satellites can be determined from the time delay of the two signals. This clever trick has the consequence that the receiver does not need to contain an atomic clock itself. To understand this better, we need to know about functions of three variables and surfaces.

Homework

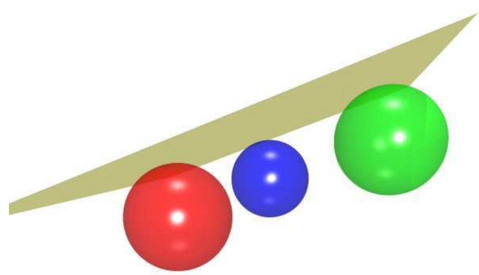
1 Find the parametric and symmetric equation for the line which passes through the points $P = (2, 3, 4)$ and $Q = (4, 5, 6)$.

2 A regular tetrahedron has vertices at the points $P_1 = (0, 0, 3)$,$P_2 = (0, \sqrt{8}, -1)$, $P_3 = (-\sqrt{6}, -\sqrt{2}, -1)$ and $P_4 = (\sqrt{6}, -\sqrt{2}, -1)$. Find the distance between two edges which do not intersect.



3 Find a parametric equation for the line through the point $P = (3, 1, 2)$ that is perpendicular to the line $L : x = 1 + t, y = 1 - t, z = 2t$ and intersects this line in a point Q.

4 Given three spheres of radius 1 centered at $A = (1, 2, 0), B = (4, 5, 0), C = (1, 3, 2)$. Find a plane $ax + by + cz = d$ which touches all of three spheres from the same side.



5 a) Find the distance between the point $P = (3, 3, 4)$ and the line $x = y = z$.
b) Parametrize the line $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ in a) and find the minimum of the function $f(t) = d(P, \vec{r}(t))^2$. Verify that the the minimal value agrees with a).