Lecture 22: Curl and Divergence

We have seen the curl in two dimensions: $\text{curl}(F) = Q_x - P_y$. By Greens theorem, it had been the average work of the field done along a small circle of radius r around the point in the limit when the radius of the circle goes to zero. Greens theorem so has explained what the curl is. In three dimensions, the curl is a vector:

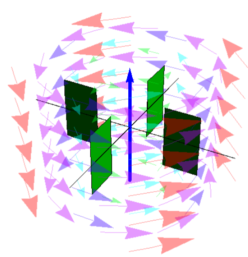
The **curl** of a vector field $\vec{F} = \langle P, Q, R\rangle$ is defined as the vector field

$$\text{curl}(P,Q,R) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle \ .$$

Invoking nabla calculus, we can write $\text{curl}(\vec{F}) = \nabla \times \vec{F}$. Note that the third component of the curl is for fixed z just the two dimensional vector field $\vec{F} = \langle P, Q\rangle$ is $Q_x - P_y$. While the curl in 2 dimensions is a scalar field, it is a vector in 3 dimensions. In n dimensions, it would have dimension $n(n-1)/2$. This is the number of two dimensional coordinate planes in n dimensions. The curl measures the "vorticity" of the field.

If a field has zero curl everywhere, the field is called **irrotational**.

The curl is often visualized using a "paddle wheel". If you place such a wheel into the field into the direction v, its rotation speed of the wheel measures the quantity $\vec{F} \cdot \vec{v}$. Consequently, the direction in which the wheel turns fastest, is the direction of $\text{curl}(\vec{F})$. Its angular velocity is the length of the curl. The wheel could actually be used to measure the curl of the vector field at any point. In situations with large vorticity like in a tornado, one can "see" the direction of the curl near the vortex center.



In two dimensions, we had two derivatives, the gradient and curl. In three dimensions, there are three fundamental derivatives, the **gradient**, the **curl** and the **divergence**.

The **divergence** of $\vec{F} = \langle P, Q, R\rangle$ is the scalar field $\text{div}(\langle P, Q, R\rangle) = \nabla \cdot \vec{F} = P_x + Q_y + R_z$.

The divergence can also be defined in two dimensions, but it is not fundamental.

The **divergence** of $\vec{F} = \langle P, Q\rangle$ is $\text{div}(P,Q) = \nabla \cdot \vec{F} = P_x + Q_y$.

In two dimensions, the divergence is just the curl of a -90 degrees rotated field $\vec{G} = \langle Q, -P\rangle$ because $\text{div}(\vec{G}) = Q_x - P_y = \text{curl}(\vec{F})$. The divergence measures the "expansion" of a field. If a field has zero divergence everywhere, the field is called **incompressible**.

With the "vector" $\nabla = \langle \partial_x, \partial_y, \partial_z\rangle$, we can write $\text{curl}(\vec{F}) = \nabla \times \vec{F}$ and $\text{div}(\vec{F}) = \nabla \cdot \vec{F}$. Formulating formulas using the "Nabla vector" and using rules from geometry is called **Nabla calculus**. This works both in 2 and 3 dimensions even so the ∇ vector is not an actual vector but an operator. The following combination of divergence and gradient often appears in physics:

$$\Delta f = \text{div}(\text{grad}(f)) = f_{xx} + f_{yy} + f_{zz} \ .$$

It is called the Laplacian of f. We can write $\Delta f = \nabla^2 f$ because $\nabla \cdot (\nabla f) = \text{div}(\text{grad}(f))$.

We can extend the Laplacian also to vector fields with

$\Delta \vec{F} = (\Delta P, \Delta Q, \Delta R)$ and write $\nabla^2 \vec{F}$.

Here are some identities:

$\text{div}(\text{curl}(\vec{F})) = 0$.
$\text{curlgrad}(\vec{F}) = \vec{0}$
$\text{curl}(\text{curl}(\vec{F})) = \text{grad}(\text{div}(\vec{F}) - \Delta(\vec{F}))$.

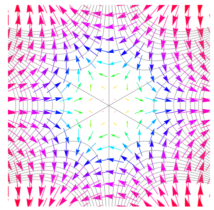
Proof. $\nabla \cdot \nabla \times \vec{F} = 0$.
$\nabla \times \nabla \vec{F} = \vec{0}$.
$\nabla \times \nabla \times \vec{F} = \nabla(\nabla \cdot \vec{F}) - (\nabla \cdot \nabla)\vec{F}$.

1 **Question:** Is there a vector field $\vec{G}$ such that $\vec{F} = \langle x + y, z, y^2\rangle = \text{curl}(\vec{G})$?
Answer: No, because $\text{div}(\vec{F}) = 1$ is incompatible with $\text{div}(\text{curl}(\vec{G})) = 0$.

2 Show that in simply connected region, every irrotational and incompressible field can be written as a vector field $\vec{F} = \text{grad}(f)$ with $\Delta f = 0$. Proof. Since $\vec{F}$ is irrotational, there exists a function f satisfying $F = \text{grad}(f)$. Now, $\text{div}(F) = 0$ implies $\text{divgrad}(f) = \Delta f = 0$.

3 Find an example of a field which is both incompressible and irrotational. Solution. Find f which satisfies the Laplace equation $\Delta f = 0$, like $f(x,y) = x^3 - 3xy^2$, then look at its gradient field $\vec{F} = \nabla f$. In that case, this gives

$$\vec{F}(x,y) = \langle 3x^2 - 3y^2, -6xy\rangle \ .$$



4 If we rotate the vector field $\vec{F} = \langle P, Q \rangle$ by 90 degrees $= \pi/2$, we get a new vector field $\vec{G} = \langle -Q, P \rangle$. The integral $\int_C F \ \cdot ds$ becomes a **flux** $\int_\gamma G \cdot dn$ of G through the boundary of R, where dn is a normal vector with length $|r'|dt$. With $\text{div}(\vec{F}) = (P_x + Q_y)$, we see that

$$\text{curl}(\vec{F}) = \text{div}(\vec{G}) \ .$$

Green's theorem now becomes

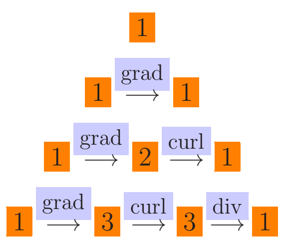
$$\int\int_R \text{div}(\vec{G}) \ dxdy = \int_C \vec{G} \cdot \vec{dn} \ ,$$

where $dn(x, y)$ is a normal vector at (x, y) orthogonal to the velocity vector $\vec{r}\,'(x, y)$ at (x, y). This new theorem has a generalization to three dimensions, where it is called Gauss theorem or divergence theorem. Don't treat this however as a different theorem in two dimensions. It is just Green's theorem in disguise.

This result shows:

The divergence at a point (x, y) is the average flux of the field through a small circle of radius r around the point in the limit when the radius of the circle goes to zero

We have now all the derivatives together. In dimension d, there are d fundamental derivatives.



They are incarnations of the same derivative, the so called **exterior derivative**.
To the end, let me stress that it is important you keep the dimensions. Many books treat two dimensional situations using terminology from three dimensions which leads to confusion. Geometry in two dimensions should be treated as a "flatlander"[1] in two dimensions only. Integral theorems become more transparent if you look at them in the right dimension. In one dimension, we had one theorem, the fundamental theorem of calculus. In two dimensions, there is the fundamental theorem of line integrals and Greens theorem. In three dimensions there are three theorems: the fundamental theorem of line integrals, Stokes theorem and the divergence theorem. We will look at the remaining two theorems next time.

[1] A. Abbott, Flatland, A romance in many dimensions,1884

Homework

1 Find a nonzero vector field $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ in each of the following cases:
a) $\vec{F}$ is irrotational but not incompressible.
b) $\vec{F}$ is incompressible but not irrotational.
c) $\vec{F}$ is irrotational and incompressible.
d) $\vec{F}$ is not irrotational and not incompressible.

The terminology in this problem comes from fluid dynamics where fluids can be incompressible, irrotational.

2 The vector field $\vec{F}(x, y, z) = \langle x, y, -2z \rangle$ satisfies $\text{div}(\vec{F}) = 0$. Can you find a vector field $\vec{G}(x, y, z)$ such that $\text{curl}(\vec{G}) = \vec{F}$? Such a field $\vec{G}$ is called a **vector potential**.
Hint. Write $\vec{F}$ as a sum $\langle x, 0, -z \rangle + \langle 0, y, -z \rangle$ and find vector potentials for each of the summand using a vector field you have seen in class.

3 Evaluate the flux integral $\int\int_S \langle 0, 0, yz \rangle \cdot \vec{dS}$, where S is the surface with parametric equation $x = uv, y = u + v, z = u - v$ on $R : u^2 + v^2 \leq 1$.

4 Evaluate the flux integral $\iint_S \text{curl}(F) \cdot \vec{dS}$ for $\vec{F}(x, y, z) = \langle xy, yz, zx \rangle$, where S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $[0, 1] \times [0, 1]$ and has an upward orientation.

5 a) What is the relation between the flux of the vector field $\vec{F} = \nabla g / |\nabla g|$ through the surface $S : \{g = 1\}$ with $g(x, y, z) = x^6 + y^4 + 2z^8$ and the surface area of S?
b) Find the flux of the vector field $\vec{G} = \nabla g \times \langle 0, 0, 1 \rangle$ through the surface S.

Remark This problem, both part a) and part do not need any computation. You can answer each question with one sentence. In part a) compare $\vec{F} \cdot \vec{dS}$ with dS in that case.