

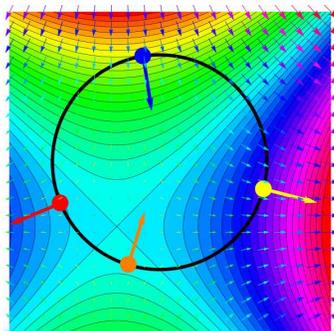
Lecture 14: Lagrange

We aim to find maxima and minima of a function $f(x, y)$ in the presence of a **constraint** $g(x, y) = 0$. A necessary condition for a critical point is that the gradients of f and g are parallel because otherwise we can move along the curve g and increase f . The directional derivative of f in the direction tangent to the level curve is zero if and only if the tangent vector to g is perpendicular to the gradient of f or if there is no tangent vector.

The system of equations $\nabla f(x, y) = \lambda \nabla g(x, y), g(x, y) = 0$, for the three unknowns x, y, λ are called **Lagrange equations**. The variable λ is a **Lagrange multiplier**.

Lagrange theorem: Extrema of $f(x, y)$ on the curve $g(x, y) = c$ are either solutions of the Lagrange equations or critical points of g .

Proof. The condition that ∇f is parallel to ∇g either means $\nabla f = \lambda \nabla g$ or $\nabla f = 0$ or $\nabla g = 0$. The case $\nabla f = 0$ can be included in the Lagrange equation case with $\lambda = 0$.



1 Minimize $f(x, y) = x^2 + 2y^2$ under the constraint $g(x, y) = x + y^2 = 1$. **Solution:** The Lagrange equations are $2x = \lambda, 4y = \lambda 2y$. If $y = 0$ then $x = 1$. If $y \neq 0$ we can divide the second equation by y and get $2x = \lambda, 4 = \lambda 2$ again showing $x = 1$. The point $x = 1, y = 0$ is the only solution.

2 Find the shortest distance from the origin to the curve $x^6 + 3y^2 = 1$. **Solution:** Minimize the function $f(x, y) = x^2 + y^2$ under the constraint $g(x, y) = x^6 + 3y^2 = 1$. The gradients are $\nabla f = \langle 2x, 2y \rangle, \nabla g = \langle 6x^5, 6y \rangle$. The Lagrange equations $\nabla f = \lambda \nabla g$ lead to the system $2x = \lambda 6x^5, 2y = \lambda 6y, x^6 + 3y^2 - 1 = 0$. We get $\lambda = 1/3, x = x^5$, so that either $x = 0$ or $x = -1$. From the constraint equation $g = 1$, we obtain $y = \sqrt{(1 - x^6)/3}$. So, we have the solutions $(0, \pm\sqrt{1/3})$ and $(1, 0), (-1, 0)$. To see which is the minimum, just evaluate f on each of the points. We see that $(0, \pm\sqrt{1/3})$ are the minima.

3 Which cylindrical soda cans of height h and radius r has minimal surface for fixed volume? **Solution:** The volume is $V(r, h) = h\pi r^2 = 1$. The surface area is $A(r, h) = 2\pi r h + 2\pi r^2$. With $x = h\pi, y = r$, you need to optimize $f(x, y) = 2xy + 2\pi y^2$ under the constrained $g(x, y) = xy^2 = 1$. Calculate $\nabla f(x, y) = \langle 2y, 2x + 4\pi y \rangle, \nabla g(x, y) = \langle y^2, 2xy \rangle$. The task is to solve $2y = \lambda y^2, 2x + 4\pi y = \lambda 2xy, xy^2 = 1$. The first equation gives $y\lambda = 2$. Putting that in the second one gives $2x + 4\pi y = 4x$ or $2\pi y = x$. The third equation finally reveals $2\pi y^3 = 1$ or $y = 1/(2\pi)^{1/3}, x = 2\pi(2\pi)^{1/3}$. This means $h = 0.54\dots, r = 2h = 1.08$. Remark: Other factors can influence the shape. For example, the can has to withstand a pressure up to 100 psi. A typical can of "Coca-Cola classic" with 3.7 volumes of CO_2 dissolve has at 75F an internal pressure of 55 psi, where PSI stands for pounds per square inch.

4 On the curve $g(x, y) = x^2 - y^3$ the function $f(x, y) = x$ obviously has a minimum $(0, 0)$. The Lagrange equations $\nabla f = \lambda \nabla g$ have no solutions. This is a case where the minimum is a solution to $\nabla g(x, y) = 0$.

Remarks.

- 1) Either of the two properties equated in the Lagrange theorem are equivalent to $\nabla f \times \nabla g = 0$ in dimensions 2 or 3.
- 2) With $g(x, y) = 0$, the Lagrange equations can also be written as $\nabla F(x, y, \lambda) = 0$ where $F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$.
- 3) Either of the two properties equated in the Lagrange theorem are equivalent to " $\nabla g = \lambda \nabla f$ or f has a critical point".
- 4) Constrained optimization problems work also in higher dimensions. The proof is the same:

Extrema of $f(\vec{x})$ under the constraint $g(\vec{x}) = c$ are either solutions of the Lagrange equations $\nabla f = \lambda \nabla g, g = c$ or points where $\nabla g = \vec{0}$.

5 Find the extrema of $f(x, y, z) = z$ on the sphere $g(x, y, z) = x^2 + y^2 + z^2 = 1$. **Solution:** compute the gradients $\nabla f(x, y, z) = \langle 0, 0, 1 \rangle, \nabla g(x, y, z) = \langle 2x, 2y, 2z \rangle$ and solve $\langle 0, 0, 1 \rangle = \nabla f = \lambda \nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z \rangle, x^2 + y^2 + z^2 = 1$. The case $\lambda = 0$ is excluded by the third equation $1 = 2\lambda z$ so that the first two equations $2\lambda x = 0, 2\lambda y = 0$ give $x = 0, y = 0$. The 4'th equation gives $z = 1$ or $z = -1$. The minimum is the south pole $(0, 0, -1)$ the maximum the north pole $(0, 0, 1)$.

6 A dice shows k eyes with probability p_k with k in $\Omega = \{1, 2, 3, 4, 5, 6\}$. A probability distribution is a nonnegative function p on Ω which sums up to 1. It can be written as a vector $(p_1, p_2, p_3, p_4, p_5, p_6)$ with $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1$. The **entropy** of the probability vector \vec{p} is defined as $f(\vec{p}) = -\sum_{i=1}^6 p_i \log(p_i) = -p_1 \log(p_1) - p_2 \log(p_2) - \dots - p_6 \log(p_6)$. Find the distribution p which maximizes entropy under the constrained $g(\vec{p}) = p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1$. **Solution:** $\nabla f = \langle -1 - \log(p_1), \dots, -1 - \log(p_n) \rangle, \nabla g = \langle 1, \dots, 1 \rangle$. The Lagrange equations are $-1 - \log(p_i) = \lambda, p_1 + \dots + p_6 = 1$, from which we get $p_i = e^{-(\lambda+1)}$. The last equation $1 = \sum_i \exp(-(\lambda+1)) = 6 \exp(-(\lambda+1))$ fixes $\lambda = -\log(1/6) - 1$ so that $p_i = 1/6$. The distribution, where each event has the same probability is the distribution of maximal entropy. Maximal entropy means **least information content**. An unfair dice allows a cheating gambler or casino to gain profit. Cheating through asymmetric weight distributions can be avoided by making the dices transparent.

7 Assume that the probability that a physical or chemical system is in a state k is p_k and that the energy of the state k is E_k . Nature tries to minimize the **free energy** $f(p_1, \dots, p_n) =$

$-\sum_i [p_i \log(p_i) - E_i p_i]$ if the energies E_i are fixed. The probability distribution p_i satisfying $\sum_i p_i = 1$ minimizing the free energy is called a **Gibbs distribution**. Find this distribution in general if E_i are given. **Solution:** $\nabla f = (-1 - \log(p_1) - E_1, \dots, -1 - \log(p_n) - E_n)$, $\nabla g = (1, \dots, 1)$. The Lagrange equation are $\log(p_i) = -1 - \lambda - E_i$, or $p_i = \exp(-E_i)C$, where $C = \exp(-1 - \lambda)$. The constraint $p_1 + \dots + p_n = 1$ gives $C(\sum_i \exp(-E_i)) = 1$ so that $C = 1/(\sum_i \exp(-E_i))$. The Gibbs solution is $p_k = \exp(-E_k)/\sum_i \exp(-E_i)$.

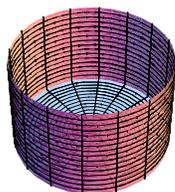
Remarks:

- 1) Can we avoid Lagrange? This is often done in single variable calculus. To extremize $f(x, y)$ under the constraint $g(x, y) = 0$ we find $y = y(x)$ from the second equation and extremize the single variable problem $f(x, y(x))$. This needs to be done carefully and the boundaries must be considered. To extremize $f(x, y) = y$ on $x^2 + y^2 = 1$ for example we need to extremize $\sqrt{1 - x^2}$. We can differentiate to get the critical points but also have to look at the cases $x = 1$ and $x = -1$, where the actual minima and maxima occur. In general also, we can not do the substitution. To extremize $f(x, y) = x^2 + y^2$ with constraint $g(x, y) = x^4 + 3y^2 - 1 = 0$ for example, we solve $y^2 = (1 - x^4)/3$ and minimize $h(x) = f(x, y(x)) = x^2 + (1 - x^4)/3$. $h'(x) = 0$ gives $x = 0$. To find the maximum $(\pm 1, 0)$, we had to maximize $h(x)$ on $[-1, 1]$, which occurs at ± 1 . To extremize $f(x, y) = x^2 + y^2$ under the constraint $g(x, y) = p(x) + p(y) = 1$, where p is a complicated function in x which satisfies $p(0) = 0, p'(1) = 2$, the Lagrange equations $2x = \lambda p'(x), 2y = \lambda p'(y), p(x) + p(y) = 1$ can be solved with $x = 0, y = 1, \lambda = 1$. We can not solve $g(x, y) = 1$ however for y in an explicit way.
- 2) How do we determine whether a solution of the Lagrange equations is a maximum or minimum? Instead of using a second derivative test, we make a list of critical points and pick the maximum and minimum. A second derivative test can be designed using second directional derivative in the direction of the tangent.
- 3) The Lagrange method also works with more constraints. With two constraints the constraint $g = c, h = d$ defines a curve in space. The gradient of f must now be in the plane spanned by the gradients of g and h because otherwise, we could move along the curve and increase f . Here is a formulation in three dimensions.

Extrema of $f(x, y, z)$ under the constraint $g(x, y, z) = c, h(x, y, z) = d$ are either solutions of the Lagrange equations $\nabla f = \lambda \nabla g + \mu \nabla h, g = c, h = d$ or solutions to $\nabla g = 0, \nabla f(x, y, z) = \mu \nabla h, h = d$ or solutions to $\nabla h = 0, \nabla f = \lambda \nabla g, g = c$ or solutions to $\nabla g = \nabla h = 0$.

Homework

- 1 Find the cylindrical basket which is open on the top has has the largest volume for fixed area π . If x is the radius and y is the height, we have to extremize $f(x, y) = \pi x^2 y$ under the constraint $g(x, y) = 2\pi xy + \pi x^2 = \pi$. Use the method of Lagrange multipliers.



- 2 Find the extrema of the same function

$$f(x, y) = e^{-x^2 - y^2} (x^2 + 2y^2)$$

as in problem 4.1.3 but now on the entire disc $\{x^2 + y^2 \leq 4\}$ of radius 2. Besides the already found extrema inside the disk, you have to find extrema on the boundary.

- 3 Find and classify all the critical points of the function

$$f(x, y) = 5 + 3x^2 + 3y^2 + y^3 + x^3.$$

Is there a global maximum or a global minimum for $f(x, y)$?

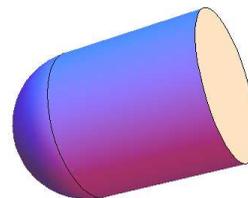
- 4 A **solid bullet** made of a half sphere and a cylinder has the volume $V = 2\pi r^3/3 + \pi r^2 h$ and surface area $A = 2\pi r^2 + 2\pi r h + \pi r^2$. Doctor Manhattan designs a bullet with fixed volume and minimal area. With $g = 3V/\pi = 1$ and $f = A/\pi$ he therefore minimizes

$$f(h, r) = 3r^2 + 2rh$$

under the constraint

$$g(h, r) = 2r^3 + 3r^2 h = 1.$$

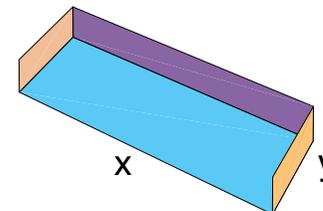
Use the Lagrange method to find a local minimum of f under the constraint $g = 1$.



- 5 Minimize the material cost of an office tray

$$f(x, y) = xy + x + 2y$$

of length x , width y and height 1 under the constraint that the volume $g(x, y) = xy$ is



constant and equal to 4.