7/22/2010 SECOND HOURLY PRACTICE IV Maths 21a, O.Knill, Summer 2010

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) T F

$(0,0)$ is a local minimum of the function $f(x,y) = x^6 + y^6$.

Solution:
$(0,0)$ is a local minimum because the value there is 0 and the function is positive.

2) T F

If $f(x,y)$ has a local max at the point $(0,0)$ with discriminant $D > 0$, then $g(x,y) = f(x,y) - x^9 + y^9$ has a local max at $(0,0)$ too.

Solution:
Adding $x^9 - y^9$ does not change the first and second derivatives.

3) T F

The value of the function $f(x,y) = \sqrt{1 + 3x + 5y}$ at $(-0.002, 0.01)$ can by linear approximation be estimated as $1 - (3/2) \cdot 0.002 + (5/2) \cdot 0.01$.

Solution:
Use the definition defining the linearization $L(x,y)$.

4) T F

The value of the double integral $\int_0^{\pi/4} \int_0^2 x^3 \cos(y) \ dxdy$ is the same as $(\int_0^2 x^3 \ dx)(\int_0^{\pi/4} \cos(y) \ dy)$.

Solution:
The function $\cos(y)$ is a constant for the inner integral so that we can pull it out of the inner integral.

5) T F

The chain rule can be written in the form $\frac{d}{dt} f(\vec{r}(t)) = D_{\vec{r}\,'(t)} f(\vec{r}(t))$

Solution:
By definition $D_v f = \nabla f \cdot f$. The chain rule is $\frac{d}{dt} f(r(t)) = \nabla f(r(t)) \cdot r'(t)$.

6) T F

The gradient of f at a point (x_0, y_0, z_0) is tangent to the level surface of f which contains (x_0, y_0, z_0).

Solution:
It is a basic and important fact that ∇f is **perpendicular** to the level surface.

7) | T | **F** |

If the Lagrange multiplier λ is positive, then the critical point under constraint is a local minimum.

Solution:
The sign of λ has nothing to say about the nature of the critical point.

8) | **T** | F |

If the directional derivative $D_{\vec{v}} f(1,1) = 0$ for all vectors $\vec{v}$, then $(1,1)$ is a critical point of $f(x,y)$.

Solution:
Especially, $D_{\nabla f}(f) = |\nabla f|^2 = 0$ so that $\nabla f = (0,0,0)$.

9) | T | **F** |

For any curve $\vec{r}(t)$, the vectors $\vec{r}\,''(t)$ and $\vec{r}\,'(t)$ are always perpendicular to each other.

Solution:
This is most of the time wrong.

10) | T | **F** |

Every critical point (x,y) of a function $f(x,y)$ for which the discriminant D is not zero is either a local maximum or a a local minimum.

Solution:
The second derivative test give for negative D that we have a saddle point.

11) | **T** | F |

The function $f(x,y) = e^y x^2 \sin(y^2)$ satisfies the partial differential equation $f_{xxyyyxyy} = 0$.

Solution:
By Clairot's theorem, we can have all three x derivatives at the beginning.

12) | **T** | F |

If $(0,0)$ is a critical point of $f(x,y)$ and the discriminant D is zero but $f_{xx}(0,0) < 0$ then $(0,0)$ can not be a local minimum.

Solution:
If $f_{xx}(0,0) < 0$ then on the x-axis the function $g(x) = f(x,0)$ has a local maximum. This means that there are points close to $(0,0)$ where the value of f is larger.

13) T F In the second derivative test, one can replace the condition $D > 0, f_{xx} > 0$ with $D > 0, f_{yy} > 0$ to check whether a point is a local minimum.

Solution:
True. If $f_{xx}f_{yy} - f_{xy}^2 > 0$, then f_{xx} and f_{yy} must have the same signs.

14) T F If $\vec{r}(t)$ is a curve in space and f is a function of three variables, then $\frac{d}{dt}f(\vec{r}(t)) = 0$ for $t = 0$ implies that $\vec{r}(0)$ is a critical point of $f(x,y)$.

Solution:
We can have $r(t) = (t,0,0)$ and $f(x,y,z) = x^2 + (y-1)^2$.

15) T F The function $f(x,y) = (x^4 - y^4)$ has neither a local maximum nor a local minimum at $(0,0)$.

Solution:
The function is both smaller and bigger than $f(0,0)$ for points near $(0,0)$.

16) T F If every point of the plane is a critical point for a function f then f is a constant function.

Solution:
Yes, $f_x = 0$ everywhere means that f is independent of x everywhere and $f_y = 0$ everywhere is equivalent that f is independent of y everywhere.

17) T F If $f(x,y)$ has a local max at the point $(0,0)$ with discriminant $D > 0$, then $g(x,y) = f(x,y) - x^9 + y^9$ has a local max at $(0,0)$ too.

Solution:
Adding $x^9 - y^9$ does not change the first and second derivatives.

18) T F

The value of the function $f(x, y) = \log(e + 3x + 5y)$ at $(-0.002, 0.01)$ can by linear approximation be estimated as $1 - 0.006 + 0.05$.

Solution:
Use the formula for $L(x, y)$.

19) T F

The chain rule tells that $\frac{d}{dt} f(\vec{r}(t)) = f(\vec{r}(t)) r'(t)$

Solution:
The gradient and dot product are missing.

20) T F

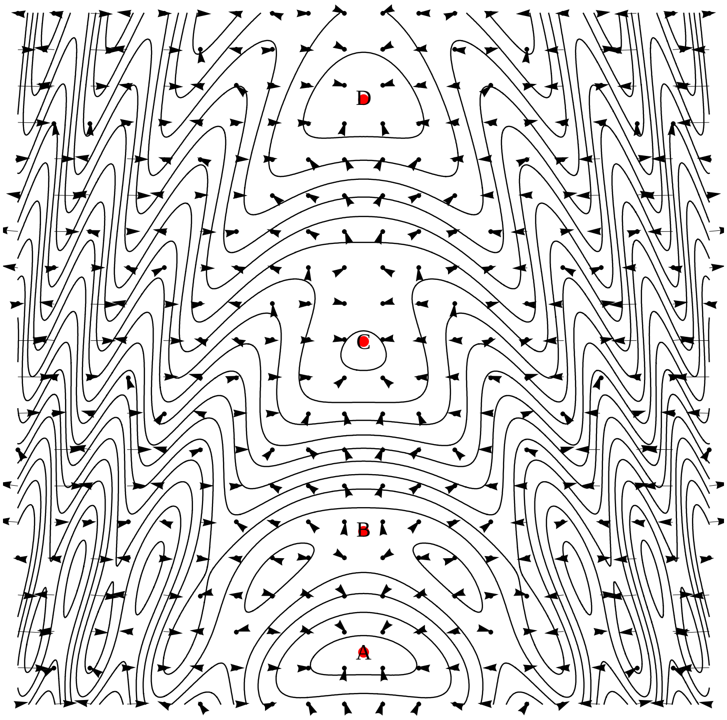
If $(0, 0)$ is a critical point of $f(x, y)$ and the discriminant D is zero but $f_{xx}(0, 0) > 0$ then $(0, 0)$ can not be a local maximum.

Solution:
If $f_{xx}(0, 0) > 0$ then on the x-axis the function $g(x) = f(x, 0)$ has a local minimum. This means that there are points close to $(0, 0)$ where the value of f is larger.

Problem 2) (10 points)

a) (5 points) The picture below shows the contour map of a function $f(x,y)$ which has many critical points. Four of them are outlined for you on the y axes and are labeled A, B, C, D and ordered in increasing y value. The picture shows also the gradient vectors. Determine from each of the 4 points whether it is a local maximum, a local minimum or a saddle point. No justification is necessary in this problem.

Point	Max	Min	Saddle
D			
C			
B			
A			



b) (5 points)

Match the integrals with those obtained by changing the order of integration. No justifications are needed. Note that one of the Roman letters I)-V) will not be used, you have to chose four out of five.

Enter I,II,III,IV or V here.	Integral
	$\int_0^1 \int_{1-y}^1 f(x,y)\ dxdy$
	$\int_0^1 \int_y^1 f(x,y)\ dxdy$
	$\int_0^1 \int_0^{1-y} f(x,y)\ dxdy$
	$\int_0^1 \int_0^y f(x,y)\ dxdy$

I) $\int_0^1 \int_0^x f(x,y)\ dydx$
II) $\int_0^1 \int_0^{1-x} f(x,y)\ dydx$
III) $\int_0^1 \int_x^1 f(x,y)\ dydx$
IV) $\int_0^1 \int_0^{x-1} f(x,y)\ dydx$
V) $\int_0^1 \int_{1-x}^1 f(x,y)\ dydx$

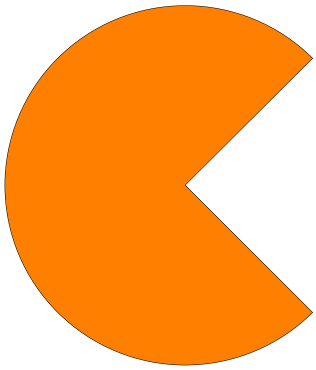
Solution:
a) D is a local maximum because the gradient arrows point towards it and gradient arrows point into the direction of maximal increase. Similarly, C and A are local maxima. The point B is a saddle point.
b) Make a picture of the region. Each of the regions is a triangle in the unit square. The first is the upper right half, the second the lower right, the third the lower left, the forth the upper left half. The solution is $\boxed{\text{V,I,II,III}}$.

Problem 3) (10 points)

The Pac-Man region R is bounded by the lines $y = x, y = -x$ and the unit circle. The number

$$a = \frac{\int \int_R x \; dxdy}{\int \int_R 1 \; dxdy}$$

defines the point $C = (a, 0)$ called center of mass of the region. Find it.



Solution:

$$\int_{\pi/4}^{7\pi/4} \int_0^1 r\cos(\theta) \; rdrd\theta = (1/3)\sin(\theta)|_{\pi/4}^{7\pi/4} = -\sqrt{2}/3 \; .$$

$$\int_{\pi/4}^{7\pi/4} \int_0^1 rdrd\theta = (1/2)(7\pi/4 - \pi/4) = 6\pi/8 = 3\pi/4 \; .$$

The second integral is the area of the Pac-Man, which is 3/4 of the area of the full disc. Dividing the first by the second integral gives the result $\boxed{a = -4\sqrt{3}/(9\pi)}$. The center of mass is $\boxed{(-4\sqrt{3}/(9\pi), 0)}$.

Problem 4) (10 points)

Find all the critical points of

$$f(x, y) = x^3 + y^3 - 3x - 12y$$

and indicate whether they are local maxima, local minima or saddle points.

Solution:
$\nabla f(x, y) = (-3 + 3x^2, -12 + 3y^2) = (0, 0)$ so that the critical points are $(0, 1), (0, -1), (1, 1), (1, -1)$. We have $D = 36xy$ and $f_{xx} = 6x$.

Point	D	f_{xx}	type
$(-1, -2)$	$D = 72$	-6	max
$(-1, 2)$	$D = -72$	6	saddle
$(1, -2)$	$D = -72$	6	saddle
$(1, 2)$	$D = 72$	6	min

Problem 5) (10 points)

When Ramanujan, an amazing mathematician who was born in India was sick in the hospital and the English mathematician Hardy visited him, Ramanujan asked "whats up?" Hardy answered. "Nothing special. Even the number of the taxi cab was boring: 1729". Ramanujan answered: "No, that is a remarkable number. It is the smallest number, which can be written in two different ways as a sum of two perfect cubes. Indeed $1729 = 1^3 + 12^3 = 9^3 + 10^3$.

a) (5 points) Find the linearization $L(x, y, z)$ of the function $f(x, y, z) = x^3 + y^3 - z^3$ at the point $(9, 10, 12)$.

b) (5 points) Use the technique of linear approximation to estimate $9.001^3 + 10.02^3 - 12.001^3$.

Solution:
We make a linear approximation of $f(x,y,z) = x^3 + y^3 - z^3$ at the point $(9,10,12)$. We have $\nabla f(x,y,z) = (3x^2, 3y^2, -3z^2)$ which is $(273, 300, 432)$. a) $L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$. b) $L(9.001, 10.02, 12.001) = 1 + 273 \cdot 0.001 + 300 \cdot 0.02 - 432 \cdot 0.001$.
To the grading: since the numerics was a bit challenging without computer, we would not have taken off any points for numerical computation errors here. We are not all Ramanujans.

Problem 6) (10 points)

a) (5 points) Find the equation $ax + by + cz = d$ for the tangent plane to the level surface

$$f(x,y,z) = x^3 + y^3 - z^3 = 1$$

at the point $(1,1,1)$. Note that this is the same Ramanujan function as in the previous problem.

b) (5 points) If we intersect the level surface $f(x,y,z) = 1$ with the plane $z = 2$, we obtain the equation for an implicit curve $x^3 + y^3 = 9$. It is a level curve for the function $g(x,y) = x^3 + y^3$. Find the tangent line to this curve at the point $(1,2)$.

Solution:
a) We have $\nabla f(1,1,1) = (3,3,-3)$ so that the plane is $3x + 3y - 3z = 3$.
b) We have $\nabla g(x,y) = (3x^2, 3y^2)$ and $\nabla g(1,2) = (3,12)$. The line has the form $3x + 12y = 27$.

Problem 7) (10 points)

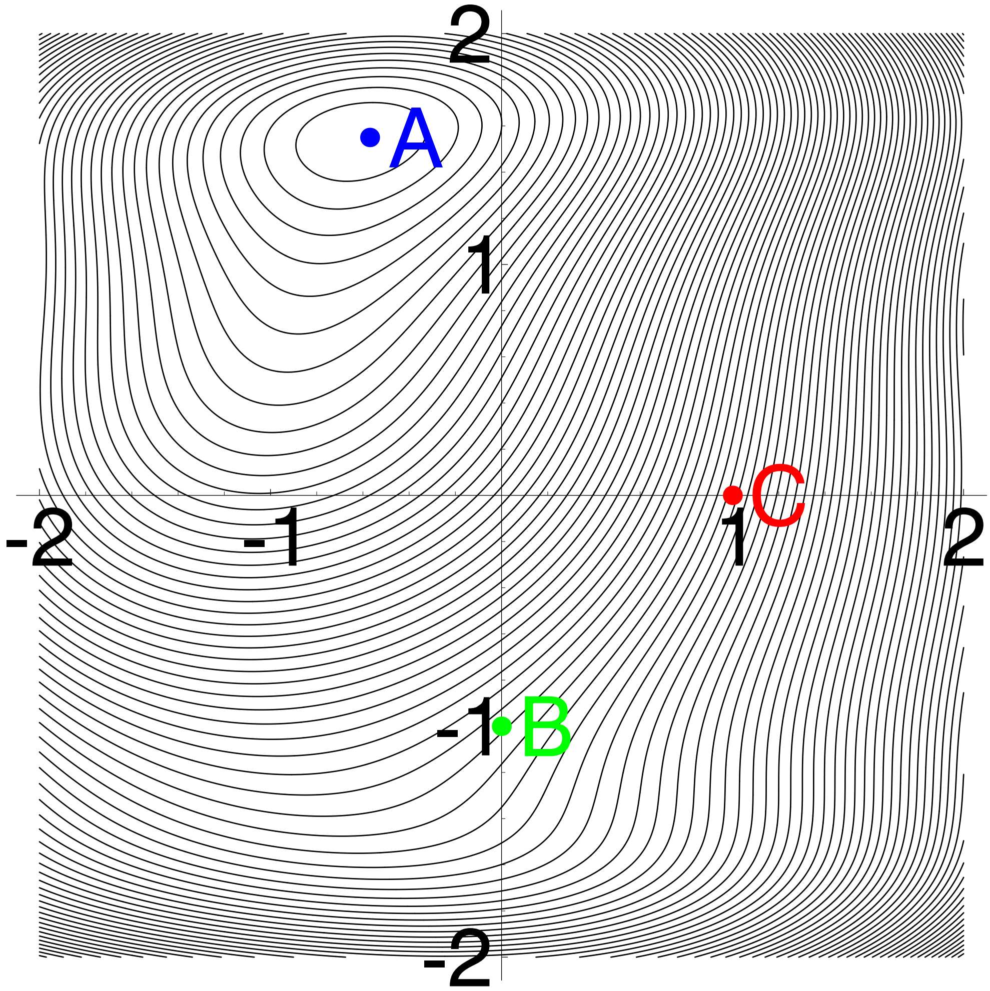
A function $f(x,y)$ of two variables describes the height of a mountain. You don't know the function but you see its level curves. The mountain has its peak at the point A on the picture.

a) (3 points) At which point does f take its maximum under the constraint $x = 0$.

b) (3 points) At which point does f take its maximum under the constraint $y = 0$.

c) (4 points) At which of the points B or C is the length of the gradient vector larger?

Note: As always, you have to give explanations to get full credit. The points in a) and b) are not necessarily marked. Give it up to an accuracy of 1/2. For example, an answer in a) or b) could look like $(0.5, 1.5)$.



Solution:
a) The point is about at $(0, 1.5)$. this is a point, where the level curve is tangent to the constrained curve $x = 0$. The fact that the point is closest to the mountain is only an accident here.
b) The point is near $(-1, 0)$. Again it is a point, where the level curve is tangent to the constrained curve $y = 0$.
c) If level curves are close to each other this means that the gradient is large and that the surface is steep at this point in the direction of the gradient. At the point C, the steepness is larger.

Problem 8) (10 points)

Olivers great-grand-dad Emil Frech Hoch (1874-1947) founded the car company "Frech-Hoch" in Switzerland. The company produces cars and trucks. The company still exists today and produces specialized vehicles.

Assume the revenue of the company is $f(x,y) = x^2 + 2y^2$, where x is the number of cars and y is the number of trucks produced per year. The production is constrained by the amount of steel available. Trucks need twice as much steel leading to $g(x,y) = x + 2y = 1$. Use the Lagrange multiplier method to find the optimal production rate.



Solution:
The Lagrange equations are

$$\begin{aligned} 2x &= \lambda \\ 4y &= 2\lambda \\ x + 2y &= 1 \end{aligned}$$

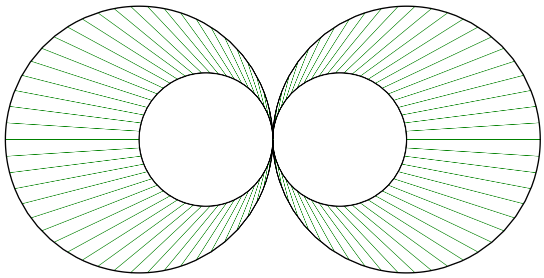
The first two equations gives $x = y$ and lead to $x = y = 1/3$. It turns out that the only solution to this Lagrange problem is $(1/3, 1/3)$. While it is an optimum it turns out be a minimum. A more realistic assumption would have been that $x^2 + 2y^2$ is a commodity which one wants to minimize, like for example production trash. Some people suggested to consider the point $(1,0)$ which is the maximum if one also has the constraints $x \geq 0$ and $y \geq 0$.

Problem 9) (10 points)

Find the area of the region in the plane given in polar coordinates by

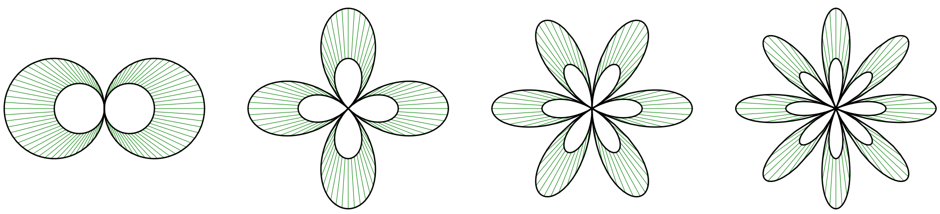
$$\{(r,\theta) \mid |\cos(\theta)| \leq r \leq 2|\cos(\theta)|, 0 \leq \theta < 2\pi \} .$$

The region is the shaded part in the figure.



Solution:
$\int_0^{2\pi} \int_{|\cos(\theta)|}^{|2\cos(\theta)|} r \; dr d\theta = \int_0^{2\pi} 4\cos^2(\theta)/2 - \cos^2(\theta)/2 \; d\theta = \boxed{3\pi/2}$. The result is the same for any region

$$\{(r, \theta) \mid |\cos(n\theta)| \leq r \leq 2|\cos(n\theta)|, 0 \leq \theta < 2\pi \} .$$



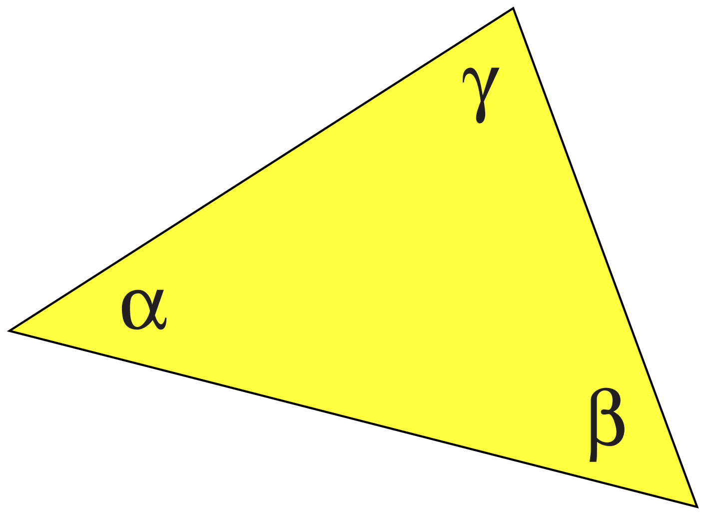
n=1 n=2 n=3 n=4

Problem 10) (10 points)

What is the shape of the triangle with angles α, β, γ for which

$$f(\alpha, \beta, \gamma) = \log\left(\sin(\alpha)\sin(\beta)\sin(\gamma)\right)$$

is maximal?



Solution:
The Lagrange equations are $\cot(\alpha) = \lambda, \cot(\beta) = \lambda, \cot(\gamma) = \lambda$. Because α, β, γ are all in $[0, \pi]$, we conclude that all are the same. From the last equation follows $\boxed{\alpha = \beta = \gamma = \pi/3}$ and $\sin(\alpha)\sin(\beta)\sin(\gamma) = (\sqrt{3}/2)^3$.