7/22/2010 SECOND HOURLY PRACTICE III Maths 21a, O.Knill, Summer 2010

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points)

1) T **F** (1, 1) is a local maximum of the function $f(x, y) = x^2y - x + \cos(y)$.

Solution:
(1, 1) is not even a critical point.

2) T **F** If f is a smooth function of two variables, then the number of critical points of f inside the unit disc is finite.

Solution:
Take $f(x, y) = x^2$ for example. Every point on the y axis $\{x = 0\}$ is a critical point.

3) T **F** The value of the function $f(x, y) = \sin(-x + 2y)$ at $(0.001, -0.002)$ can by linear approximation be estimated as -0.003.

Solution:
The correct approximation would be $f(0, 0) + 0.001(-1) - 0.002(2) = -0.005$.

4) **T** F If (1, 1) is a critical point for the function $f(x, y)$ then (1, 1) is also a critical point for the function $g(x, y) = f(x^2, y^2)$.

Solution:
If $\nabla f(1, 1) = (f_x(1, 1), f_y(1, 1)) = (0, 0)$ then also $\nabla g(1, 1) = (f_x(1, 1)2x, f_y(1, 1)2y) = (0, 0)$. Note that the statement would not be true, if we would replace (1, 1) say with (2, 2) (as in the practice exam).

5) **T** F If the velocity vector $\vec{r}\,'(t)$ of the planar curve $\vec{r}(t)$ is orthogonal to the vector $\vec{r}(t)$ for all times t, then the curve is a circle.

Solution:
$d/dt(\vec{r}(t) \cdot \vec{r}(t)) = \vec{r}(t) \cdot \vec{r}'(t) = 0$ by assumption. This shows that $|\vec{r}(t)|$ is constant.

6) **T** F The gradient of $f(x, y)$ is normal to the level curves of f.

Solution:
This is a basic and important fact.

7) | T | **F** |

If (x_0, y_0) is a maximum of $f(x, y)$ under the constraint $g(x, y) = g(x_0, y_0)$, then (x_0, y_0) is a maximum of $g(x, y)$ under the constraint $f(x, y) = f(x_0, y_0)$.

Solution:
Assume you have a situation f, g, where this is true and where the constraint is $g = 0$, produce a new situation $f, h = -g$, where the first statement is still true but where the extrema of h under the constraint of f is a minimum.

8) | **T** | F |

If $\vec{u}$ is a unit vector tangent at (x, y, z) to the level surface of $f(x, y, z)$ then the directional derivative satisfies $D_u f(x, y, z) = 0$.

Solution:
The directional derivative measures the rate of change of f in the direction of u. On a level surface, in the direction of the surface, the function does not change (because f is constant by definition on the surface).

9) | T | **F** |

If $\vec{r}(t) = \langle x(t), y(t) \rangle$ and $x(t), y(t)$ are polynomials, then the tangent line is defined at all points.

Solution:
Take the example $\vec{r}(t) = \langle t^2, t^3 \rangle$. At $t = 0$, we have a cusp and the gradient is zero. We do not have a tangent line there.

10) | **T** | F |

The vector $\vec{r}_u(u, v)$ is tangent to the surface parameterized by $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$.

Solution:
The vector $\vec{r}_u$ is tangent to a grid curve and so tangent to the surface.

11) | T | **F** |

The second derivative test allows to check whether an extremum found with the Lagrange multiplier method is a maximum.

Solution:
No, the second derivative test applies for function $f(x, y)$ without constraint.

12) T F

If $(0, 0)$ is a critical point of $f(x, y)$ and the discriminant D is zero but $f_{xx}(0, 0) > 0$ then $(0, 0)$ can not be a local maximum.

Solution:
If $f_{xx}(0, 0) > 0$ then on the x-axis the function $g(x) = f(x, 0)$ has a local minimum. This means that there are points close to $(0, 0)$ where the value of f is larger.

13) T F

Let (x_0, y_0) be a saddle point of $f(x, y)$. For any unit vector $\vec{u}$, there are points arbitrarily close to (x_0, y_0) for which ∇f is parallel to $\vec{u}$.

Solution:
Just look at the level curves near a saddle point. The gradient vectors are orthogonal to the level curves which are hyperbola. You see that they point in any direction except 4 directions. To see this better, take a pen and draw a circle around the saddle point between two of your knuckles on your fist. At each point of the circle, now draw the direction of steepest increase (this is the gradient direction).

14) T F

If $f(x, y)$ has two local maxima on the plane, then f must have a local minimum on the plane.

Solution:
Look at a camel type surface. There is no local minimum.

15) T F

Given a unit vector v, define $g(x) = D_v f(x)$. If at a critical point, for all vectors v we have $D_v g(x) > 0$, then f is a local maximum.

Solution:
On every line through the critical point, we have a local minumum. So, it is a local minimum, not a local maximum

16) T F

If $x^4 y + \sin(y) = 0$ then $y' = 4x^3/(x^4 + \cos(y))$.

Solution:
The sign is wrong.

17) T F

The critical points of $F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$ are solutions to the Lagrange equations when extremizing the function $f(x, y)$ under the constraint $g(x, y) = 0$.

Solution:
The critical points of F are points where $f_x = \lambda g_x, f_y = \lambda g_y, g = 0$ which is exactly the Lagrange equations.

18) | T | F |

The volume under the graph of $f(x,y) = x^2+y^2$ inside the cylinder $x^2+y^2 \leq 1$ is $\int_0^1 \int_0^{2\pi} r^3 \, d\theta dr$.

Solution:
$x^2 + y^2 = r^2$.

19) | T | F |

The surface area of the unit sphere is 4π.

Solution:
We computed that in class.

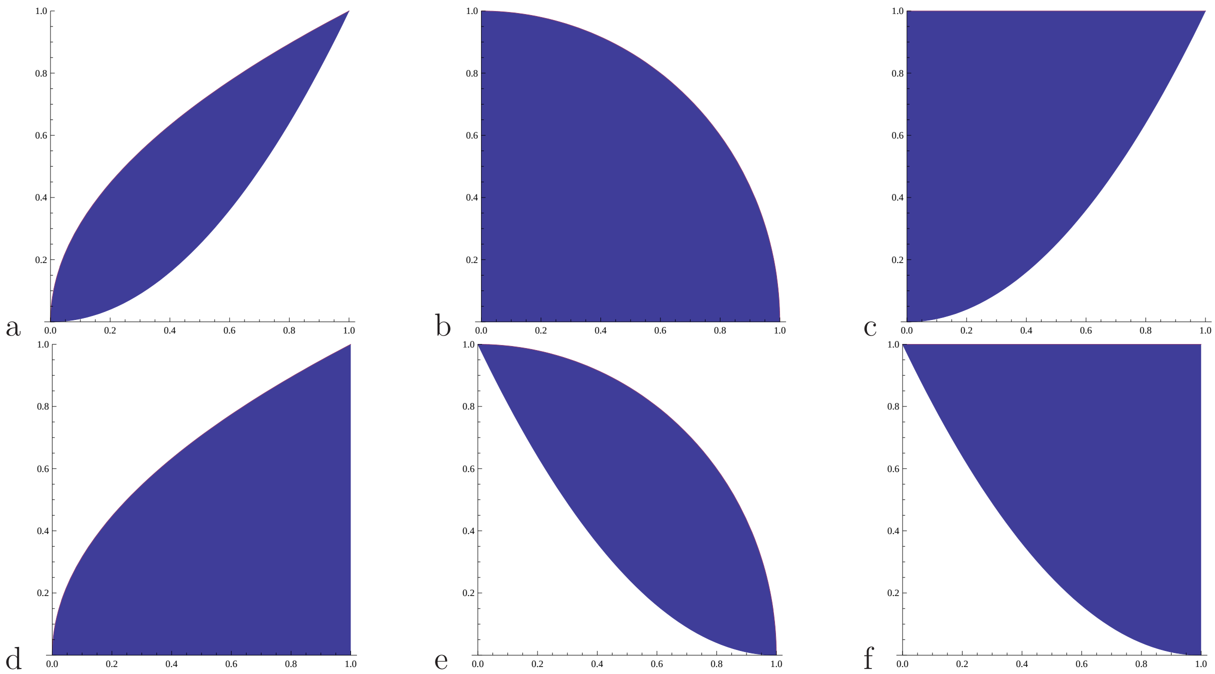
20) | T | F |

The area of a disc of radius $2r$ is 4 times larger than a disc of radius r.

Solution:
We know the area to be πr^2.

Problem 2) (10 points)

Match the regions with the corresponding double integrals.



Enter a,b,c,d,e or f	Integral of $f(x,y)$
	$\int_0^1 \int_{x^2}^{\sqrt{x}} f(x,y)\ dydx$
	$\int_0^1 \int_0^{\sqrt{y}} f(x,y)\ dxdy$
	$\int_0^1 \int_{y^2}^1 f(x,y)\ dxdy$

Enter a,b,c,d,e or f	Integral of $f(x,y)$
	$\int_0^1 \int_0^{\sqrt{1-x^2}} f(x,y)\ dydx$
	$\int_0^1 \int_{(1-x)^2}^1 f(x,y)\ dydx$
	$\int_0^1 \int_{(1-x)^2}^{\sqrt{1-x^2}} f(x,y)\ dydx$

Solution:
a),b),c),f),d),e).

Problem 3) (10 points)

a) Use the technique of linear approximation to estimate $f(\log(2)+0.001, 0.006)$ for $f(x,y) = e^{2x-y}$. (Here, log means the natural logarithm).
b) Find the equation $ax + by = d$ for the tangent line which goes through the point $(\log(2), 0)$.

Solution:
a) $L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$
$f(x_0,y_0) = e^{2\log 2} = 4$
$f_x(x_0,y_0) = 8$
$f_y(x_0,y_0) = -4$
$L(x,y) = 4 + 0.001 \cdot 8 - 4 \cdot 0.006 = \boxed{3.984}$.
b) We have $a = 8$ and $b = -4$ and get $d = 8\log(2)$ so that the line has the equation $\boxed{8x - 4y = 8\log(2)}$.

Problem 4) (10 points)

Find a point on the surface $g(x,y,z) = \frac{1}{x} + \frac{1}{y} + \frac{8}{z} = 1$ for which the distance to the origin is a local minimum.

Solution:
This is a Lagrange problem. One wants to minimize $f(x,y,z) = x^2 + y^2 + z^2$ under the constraint $g(x,y,z) = 1$. The Lagrange equations are

$$\begin{aligned} \frac{-1}{x^2} &= 2\lambda x \\ \frac{-1}{y^2} &= 2\lambda y \\ \frac{-8}{z^2} &= 2\lambda z \\ \frac{1}{x} + \frac{1}{y} + \frac{8}{z} &= 1 \end{aligned}$$

The first two equations show $x = y$, the first and third equations show $8/z^3 = 1/x^3$ or $z = 2x$. Plugging this into the last equation gives $2/x + 8/(2x) = 1$ or $x = 6, y = 6, z = 12$. $\boxed{(x,y,z) = (6,6,12)}$.
The global picture is interesting: consider the points $(x,y,z) = (1,-1/n,8/n)$, where n is a large integer, One can check that these points ly on the surface $g(x,y,z) = 1$. Their distance to the origin decreases to 1 if n goes to infinity. So the point $(6,6,12)$, while a local minimum is not a global minimum.

Problem 5) (10 points)

Find all extrema of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$ on the plane and characterize them. Do you find a absolute maximum or absolute minimum amoung them?

Solution:
The critical points satisfy $\nabla f(x,y) = (0,0)$ or $(3x^2-3, 3y^2-12) = (0,0)$. There are 4 critical points $(x,y) = (\pm 1, \pm 2)$. The discriminant is $D = f_{xx}f_{yy} - f_{xy}^2 = 36xy$ and $f_{xx} = 6x$.

point	D	f_{xx}	classification	value
(-1,-2)	72	-6	maximum	38
(-1, 2)	-72	-6	saddle	6
(1, -2)	-72	6	saddle	34
(1, 2)	72	6	minimum	2

Note that there are no global (= absolute) maxima nor global minima because the function takes arbitrarily large and small values. For $y = 0$ the function is $g(x) = f(x,0) = x^3 - 3x + 20$ which satisfies $\lim_{x\to\pm\infty} g(x) = \pm\infty$.

Problem 6) (10 points)

Find the surface area of the ellipse cut from the plane $z = 2x+2y+1$ by the cylinder $x^2+y^2 = 1$.

Solution:
Parametrize the surface $r(u,v) = \langle u, v, 2u+2v+1\rangle$ on the disc $R = \{u^2+y^2 \leq 1\}$. We get $r_u \times r_v = \langle 1,0,2\rangle \times \langle 0,1,2\rangle = \langle -2,-2,1\rangle$ and $|r_u \times r_v| = 3$. The surface integral $\int\int_R |r_u \times r_v| \; dudv = \int\int_R 3 \; dudv = 3\int\int_R \; dudv$ which is 3 times the area of the disc R: Solution: $\boxed{3\pi}$.

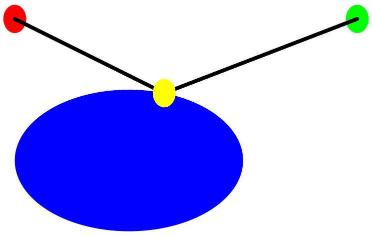
Problem 7) (10 points)

Find the tangent plane to the surface $f(x,y,z) = x^3y - xy^2 + 3z = 6$ at the point $(1,1,2)$.

Solution:
The gradient of f is $\nabla f(x,y,z) = (3x^2y - y^2, x^3 - 2xy, 3)$ which is at the point $(1,1,2)$ equal to $\nabla f(1,1,2) = (2,-1,3)$. The plane has therefore the equation $2x - y + 3z = d$. The constant d can be obtained from plugging in the coordinates of the point $d = 7$. The final answer is $2x - y + 3z = 7$.

Problem 8) (10 points)

You find yourself in the desert at the point $A = (a, 1)$, completely dehydrated and almost dead. You want to reach the point $B = (b, 1)$ as fast as possible but you can not reach it without water. There is an lake inside the ellipsoid $g(x, y) = x^2 + 2y^2 = 1$. The amount of "effort" you need to go from a point (x, y) to a point (u, v) is assumed to be $(x - u)^2 + (y - v)^2$ (this is justified by the fact that if you walk for a long time, you walk less and less efficiently so that walking twice as long will take you 4 times as much effort). Find the path of least effort which connects A with $X = (x, y)$ and then with B.



a) Which function $f(x, y)$ do you extremize? The parameters a, b are constants.
b) Write down the Lagrange equations.
c) Solve the Lagrange equations in the case $a = -1, b = 1$.

Solution:
a) We have to extremize $f(x, y) = (x - a)^2 + (y - 1)^2 + (x - b)^2 + (y - 1)^2$ under the constraint $x^2 + 2y^2 = 1$.
b) The Lagrange equations are

$$\begin{aligned} 2(x-a) + 2(x-b) &= 2\lambda x \\ 4(y-1) &= 4\lambda y \\ x^2 + 2y^2 &= 1 \end{aligned}$$

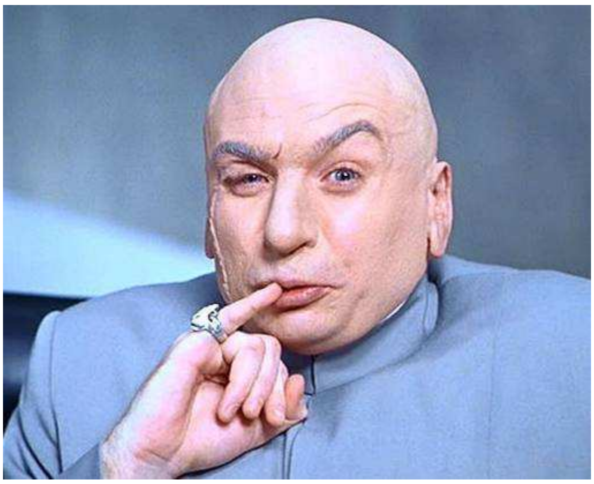
c) In the case $a = -1, b = 1$ we have extremal solutions $(0, 1/\sqrt{2})$ and $(0, -1/\sqrt{2})$. The first one is the minimum, the second the maximum.

Problem 9) (10 points)

a) (5 points) Integrate $f(x, y) = x^2 - y^2$ over the unit disk $\{x^2 + y^2 \leq 1\}$.

b) (5 points) An evil integral!

$$\int_0^1 \int_0^{\sqrt{1-\theta^2}} r^2 \, dr d\theta \; .$$



Solution:
a) Use polar coordinates:

$$\int_0^1 \int_0^{2\pi} (r^2 \cos^2(\theta) - r^2 \sin^2(\theta)) r \, d\theta dr = \int_0^1 r^3 \, dr (\int_0^{2\pi} \cos(2\theta) \, d\theta) = (1/4) \cdot 0 = 0 \, .$$

The final answer is $\boxed{\text{zero}}$.
b) Write it in more convenient coordinates:

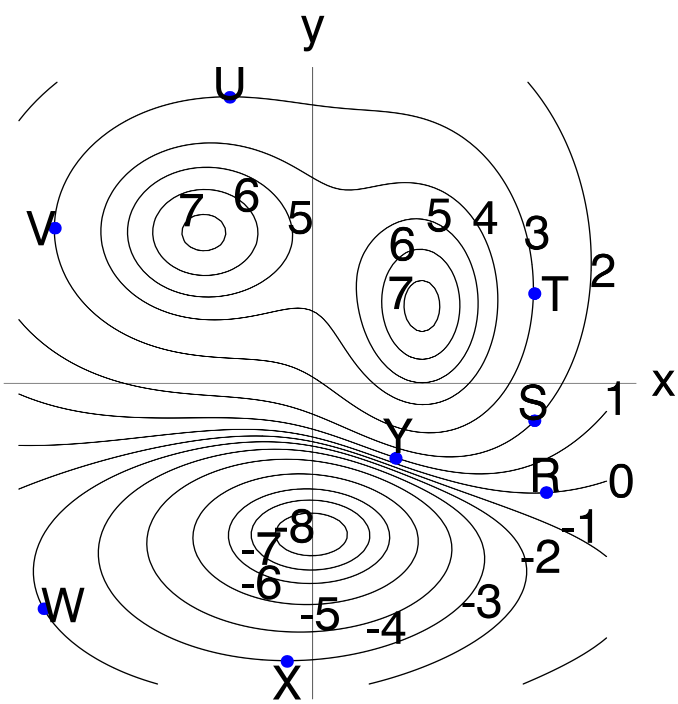
$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dydx \, .$$

This is a quarter disc in the x, y plane. Now use polar coordinates. The integral is evil because now, the θ, r have a different meaning. The integral in polar coordinates is

$$\int_0^1 \int_0^{\pi/2} r^2 \sin^2(\theta) r \, d\theta dr$$

which is $\int_0^{\pi/2} \sin^2(\theta) \, d\theta \int_0^1 r^3 \, dr = (\pi/4)(1/4) = \pi/16$. (to compute the first integral, use the double angle formula $(1 - \cos(2\theta))/2 = \sin^2(\theta)$.)

Problem 10) (10 points)



a) (4 points) Circle the point at which the magnitude of the gradient vector ∇f is greatest. Mark exactly one point. Justify your answer.

R	S	T	U	V	W	X	Y

b) (3 points) Circle the points at which the partial derivative f_x is strictly positive. Mark any number of points on this question. Justify your answers.

R	S	T	U	V	W	X	Y

c) (3 points) We know that the directional derivative in the direction $(1,1)/\sqrt{2}$ is zero at one of the following points. Which one? Mark exactly one point on this question.

R	S	T	U	V	W	X	Y

Solution:
a) At the point Y the level curves are closest to each other indicating the steepest place and so the largest gradient.
b) At the points V and Y, the function increases, if we go into the x direction. In the other points, the function decreases, if we go into the x direction.
c) In order to have a zero directional derivative, we need the gradient to be zero or perpendicular into the direction $\vec{v}$. This is the case at the point S.