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Chapter 5. Triple integrals and vector fields

Section 5.1: Triple integrals

If $f(x,y,z)$ is a function of three variables and E is a **solid region** in space, then $\int\int\int_E f(x,y,z)\ dxdydz$ is defined as the $n \to \infty$ limit of the Riemann sum $\frac{1}{n^3}\sum_{(x_i,y_j,z_k)\in E} f(x_i,y_j,z_k)$, where $(x_i,y_j,z_k) = (\frac{i}{n},\frac{j}{n},\frac{k}{n})$.

As in two dimensions, triple integrals can be evaluated by iterated 1D integral computations. Here is a simple example:

Assume E is the box $[0,1]\times[0,1]\times[0,1]$ and $f(x,y,z) = 24x^2y^3z$.

$$\int_0^1 \int_0^1 \int_0^1 24x^2y^3z\ dz\ dy\ dx .$$

To compute the integral we start from the core $\int_0^1 24x^2y^3z\ dz = 12x^3y^3$, then integrate the middle layer, $\int_0^1 12x^3y^3\ dy = 3x^2$ and finally and finally handle the outer layer: $\int_0^1 3x^2 dx$ =1.

When we calculate the most inner integral, we fix x and y. The integral is integrating up $f(x,y,z)$ along a line intersected with the body. After completing the middle integral, we have computed the integral on the plane $z = const$ intersected with R. The most outer integral sums up all these two dimensional sections.

A special case of a triple integral is the volume under the graph of a function $f(x,y)$ and above a region $R = [a,b]\times[c,d]$ is the integral $\int_a^b \int_c^d f(x,y)\ dxdy$. What we actually have computed is a triple integral

$$V = \int_a^b \int_c^d \int_0^{f(x,y)} 1\ dzdxdy .$$

Similarly, when we computed the area of a region R, we had $\int\int_R 1\ dA$, we have the volume of a solid E as $\int\int\int_E 1\ dV$.
Example: The volume of a sphere is

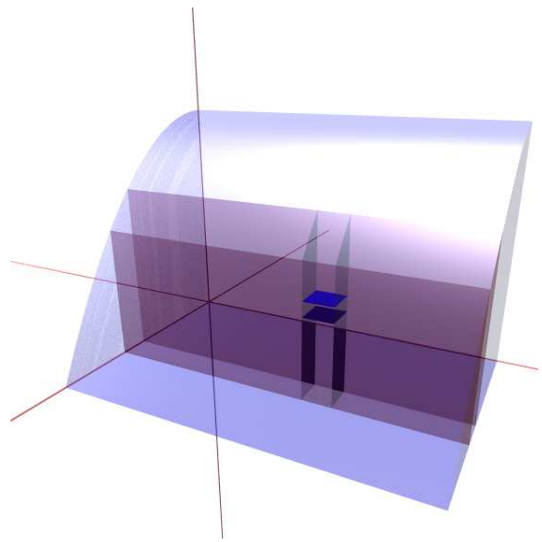
$$V = \int\int\int_R dxdydz = \int_{-1}^1 [\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz]dy]dx .$$

After computing the inner integral, we have $V = 2\int_{-1}^1 [\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-x^2-y^2)^{1/2} dy]dx$.
To resolve the next layer, call $1-y^2 = a^2$. The task is to find $\int_{-a}^a \sqrt{a^2-y^2}\ dy$. Make the substitution $y/a = \sin(u), dy = a\cos(u)$ to write this as $a\int_0^{\arcsin(a/a)} \sqrt{1-\sin^2(u)} a\cos(u) du = a^2 \int_0^{\pi/2} \cos^2(u) du = a^2\pi/2$. We can finish up the last integral

$$V = 2\pi/2 \int_{-1}^1 (1-x^2)\ dx = 4\pi/3 .$$

In general, the mass of a body with density $\rho(x,y,z)$ is $\int\int\int_R \rho(x,y,z)\ dV$. For bodies with constant density ρ the mass is ρV, where V is the volume. Compute the mass of a body which is bounded by the parabolic cylinder $z = 4-x^2$, and the planes $x = 0, y = 0, y = 6, z = 0$ if the density of the body is 1.

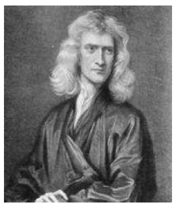
$$\int_0^2 \int_0^6 \int_0^{4-x^2} dz\ dy\ dx = \int_0^2 \int_0^6 (4-x^2)\ dydx$$
$$= 6\int_0^2 (4-x^2)\ dx = 6(4x - x^3/3)|_0^2 = 32$$



The problem of computing volumes has been tackled early in mathematics:

Archimedes (287-212 BC) designed a method of integration which allowed him to find areas, volumes and surface areas in many cases without calculus. His method of **exhaustion** is close to the numerical method of integration by Riemann sum. The **Archimedes principle** states that any body submerged in a water is acted upon by an upward force which is equal to the weight of the displaced water. This provides a practical way to compute volumes of complicated bodies. Heureka!

Cavalieri (1598-1647) would determine area and volume using tricks like the **Cavalieri principle**. Example: to get the volume the half sphere of radius R, cut away a cone of height and radius R from a cylinder of height R and radius R. At height z, this body has a cross section with area $R^2\pi - r^2\pi$. If we cut the half sphere at height z, we obtain a disc of area $(R^2 - r^2)\pi$. Because these areas are the same, the volume of the half-sphere is the same as the cylinder minus the cone: $\pi R^3 - \pi R^3/3 = 2\pi R^3/3$ and the volume of the sphere is $4\pi R^3/3$.



Newton (1643-1727) and Leibniz(1646-1716): Newton and Leibniz, developed calculus independently. The new tool made it possible to compute integrals through "anti-derivation". Suddenly, it became possible to find integrals using analytic tools as we do here.

Here is an other way to compute integrals: Suppose we want to calculate the volume of some solid body R which we assumed to be contained inside the unit cube $[0,1] \times [0,1] \times [0,1]$. The **Monte Carlo method** shoots randomly n times onto the unit cube and count the number k of times, we hit the solid. The result k/n estimates of the volume. Here is an experiment with Mathematica and where the body is one eights of the unit ball:

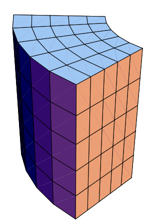
$$\mathrm{R := Random[]; k = 0; Do[x = R; y = R; z = R; If[x^2 + y^2 + z^2 < 1, k++], \{10000\}]; k/10000}$$

Assume, we hit 5277 of n=10000 times. The volume so measured is 0.5277. The actual volume of 1/8'th of the sphere is $\pi/6 = 0.524$. For $n \to \infty$ the Monte Carlo computation gives the actual volume. The Monte-Carlo integral is stronger than the Riemann integral. It is equivalent to the **Lebesgue integral** and allows to measure much more sets than solids with piecewise smooth boundaries.

Section 5.2: Spherical and cylindrical coordinates

Cylindrical coordinates are obtained by taking polar coordinates in the x-y plane and leave the z-coordinate. With $T(r,\theta,z) = (r\cos(\theta), r\sin(\theta), z)$, the integration factor r is the same as in polar coordinates.

$$\iint_{T(R)} f(x,y,z)\ dxdydz = \iint_R g(r,\theta,z) \boxed{r}\ drd\theta dz$$

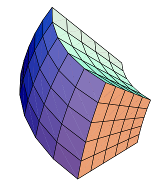


Spherical coordinates use ρ, the distance to the origin as well as two angles: θ the polar angle and ϕ, the angle between the vector and the z axis. The coordinate change is

$$T : (x,y,z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi)) .$$

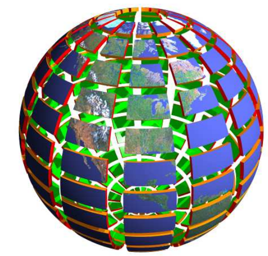
The integration factor can be seen by measuring the volume of a **spherical wedge** which is $d\rho, \rho\sin(\phi)\ d\theta, \rho d\phi = \rho^2\sin(\phi) d\theta d\phi d\rho$.

$$\iint_{T(R)} f(x,y,z)\ dxdydz = \iint_R g(\rho,\theta,z) \boxed{\rho^2\sin(\phi)}\ d\rho d\theta d\phi$$



Example: A sphere of radius R has the volume

$$\int_0^R \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin(\phi)\ d\phi d\theta d\rho .$$

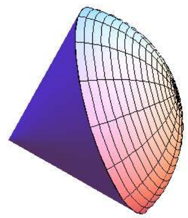


The most inner integral $\int_0^\pi \rho^2\sin(\phi) d\phi = -\rho^2\cos(\phi)|_0^\pi = 2\rho^2$. The next layer is, because ϕ does not appear: $\int_0^{2\pi} 2\rho^2\ d\phi = 4\pi\rho^2$. The final integral is $\int_0^R 4\pi\rho^2\ d\rho = 4\pi R^3/3$.

The moment of inertia of a body G with respect to an axis L is defined as the triple integral $\int\int\int_G r(x,y,z)^2\ dzdydx$, where $r(x,y,z) = R\sin(\phi)$ is the distance from the axis L. For a sphere of radius R we obtain with respect to the z-axis:

$$\begin{aligned} I &= \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho^2\sin^2(\phi)\rho^2\sin(\phi)\ d\phi d\theta d\rho \\ &= (\int_0^\pi \sin^3(\phi)\ d\phi)(\int_0^R \rho^4\ dr)(\int_0^{2\pi}\ d\theta) \\ &= (\int_0^\pi \sin(\phi)(1-\cos^2(\phi))\ d\phi)(\int_0^R \rho^4\ dr)(\int_0^{2pi}\ d\theta) \\ &= (-\cos(\phi) + \cos(\phi)^3/3)|_0^\pi (L^5/5)(2\pi) = \frac{4}{3}\cdot\frac{R^5}{5}\cdot 2\pi = \frac{8\pi R^5}{15} . \end{aligned}$$

If the sphere rotates with angular velocity ω, then $I\omega^2/2$ is the **kinetic energy** of that sphere. **Example:** the moment of inertia of the earth is $8 \cdot 10^{37} kgm^2$. The angular velocity is $\omega = 2\pi/day = 2\pi/(86400s)$. The rotational energy is $8 \cdot 10^{37} kgm^2/(7464960000s^2) \sim 10^{29} J \sim 2.510^{24} kcal$. How long would you have to run on a tread mill to accumulate this energy if you could make 2'500 kcal/hour? We would have to run 10^{21} hours $= 3.610^{24}$ seconds. Note that the universe is about 10^{17} seconds old. If all the 6 million people in Massachusetts would have run since the big bang on a treadmill, they could have produced the necessary energy to bring the earth to the current rotation. To make classes pass faster, we need to spin the earth more and just to add some more treadmills ... To the right you see my proposal for the science center.

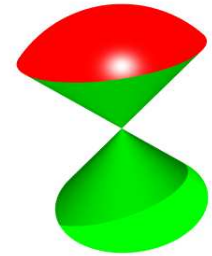


Example: Find the volume and the center of mass of a diamond, the intersection of the unit sphere with the cone given in cylindrical coordinates as $z = \sqrt{3}r$.

Solution: we use spherical coordinates to find the center of mass

$$\begin{aligned}
V &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^2 \sin(\phi)\, d\phi d\theta d\rho = \frac{(1-\frac{\sqrt{3}}{2})}{3} 2\pi \\
\overline{x} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \sin^2(\phi) \cos(\theta)\, d\phi d\theta d\rho \frac{1}{V} = 0 \\
\overline{y} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \sin^2(\phi) \sin(\theta)\, d\phi d\theta d\rho \frac{1}{V} = 0 \\
\overline{z} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \cos(\phi) \sin(\phi)\, d\phi d\theta d\rho \frac{1}{V} = \frac{2\pi}{32V}
\end{aligned}$$

Example: Find $\int\int\int_R z^2\, dV$ for the solid obtained by intersecting $\{0 \leq x^2+y^2+z^2 \leq 4\}$ with the double cone $\{z^2 \geq x^2+y^2\}$.



Solution: since the result for the double cone is twice the result for the single cone, we work with the diamond shaped region R in $\{z > 0\}$ and multiply the result at the end with 2.

In spherical coordinates, the solid R is given by $1 \leq \rho \leq 2$ and $0 \leq \phi \leq \pi/4$. With $z = \rho\cos(\phi)$, we have

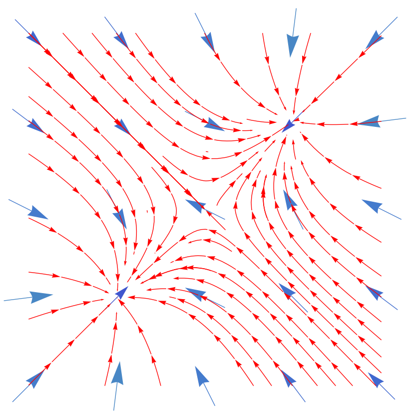
$$\int_1^2 \int_0^{2\pi} \int_0^{\pi/4} \rho^4 \cos^2(\phi) \sin(\phi)\, d\phi d\theta d\rho$$

$$= (\frac{2^5}{5} - \frac{1^5}{5}) 2\pi (\frac{-\cos^3(\phi))}{3}|_0^{\pi/4} = 2\pi \frac{31}{5}(1 - 2^{-3/2}) \; .$$

The result for the double cone is $\boxed{4\pi(31/5)(1-1/\sqrt{2}^3)}$.

Section 5.3: Vector fields

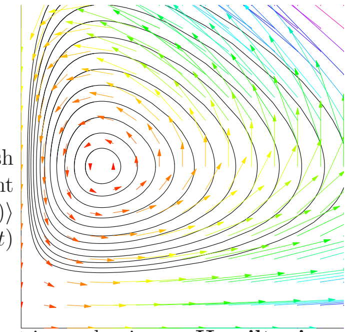
A **vector field** in the plane is a map, which assigns to each point (x, y) in the plane a vector $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$.
A vector field in space is a map, which assigns to each point (x, y, z) in space a vector $\vec{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$.
For example $\vec{F}(x, y) = \langle x-1, y\rangle/((x-1)^2 + y^2)^{3/2} - \langle x+1, y\rangle/((x+1)^2 + y^2)^{3/2}$ is the electric field of positive and negative point charge. It is called **dipole field**. It is shown in the picture above.



Gradient fields. An important class of vector fields are gradient fields. Here is the definition. If $f(x, y)$ is a function of two variables, then $\vec{F}(x, y) = \nabla f(x, y)$ is called a **gradient field**. Gradient fields in space are of the form $\vec{F}(x, y, z) = \nabla f(x, y, z)$.
When is a vector field a gradient field? $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle = \nabla f(x, y)$ implies $Q_x(x, y) = P_y(x, y)$. If this does not hold at some point, F is no gradient field. We will see next week that the condition $\text{curl}(F) = Q_x - P_y = 0$ is also necessary for F to be a gradient field. In class, we see more examples on how to construct the potential f from the gradient field F.

Vector fields and differential equations Example: Let $x(t)$ denote the population of a "prey species" like tuna fish and $y(t)$ is the population size of a "predator" like sharks. We have $x'(t) = ax(t) + bx(t)y(t)$ with positive a, b because both more predators and more prey species will lead to prey consumption. The rate of change of $y(t)$ is $-cy(t) + dxy$, where c, d are positive. We have a negative sign in the first part because predators would die out without food. The second term is explained because both more predators as well as more prey leads to a growth of predators through reproduction. A concrete example is the **Volterra-Lodka system**

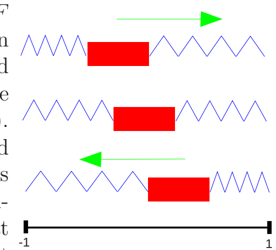
$$\begin{aligned}\dot{x} &= 0.4x - 0.4xy \\ \dot{y} &= -0.1y + 0.2xy\end{aligned}$$



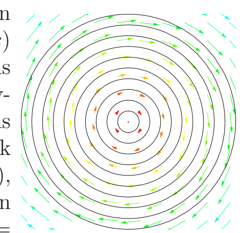
Volterra explained with such systems the oscillation of fish populations in the Mediterranean sea. At any specific point $\vec{r}(x, y) = \langle x(t), y(t)\rangle$, there is a curve $= \vec{r}(t) = \langle x(t), y(t)\rangle$ through that point for which the tangent $\vec{r}\,'(t) = (x'(t), y'(t)$ is the vector $\langle 0.4x - 0.4xy, -0.1y + 0.2xy\rangle$.

Hamiltonian fields. An other class of vector fields important in mechanics are **Hamiltonian fields**: If $H(x, y)$ is a function of two variables, then $\langle H_y(x, y), -H_x(x, y)\rangle$ is called a **Hamiltonian vector field**. An example is the harmonic Oscillator $H(x, y) = x^2 + y^2$. Its vector field $(H_y(x, y), -H_x(x, y)) = (y, -x)$ is the same as in example 1) above. You have seen earlier that the flow lines of a Hamiltonian vector fields are located on the level curves of H.

Vector fields in mechanics Newton's law $m\vec{r}'' = F$ relates the acceleration $\vec{r}''$ of a body with the force F acting at the point. For example, if $x(t)$ is the position of a mass point in $[-1, 1]$ attached at two springs and the mass is $m = 2$, then the point experiences a force $(-x + (-x)) = -2x$ so that $mx'' = 2x$ or $x''(t) = -x(t)$. If we introduce $y(t) = x'(t)$ of t, then $x'(t) = y(t)$ and $y'(t) = -x(t)$. Of course y is the velocity of the mass point, so a pair (x, y), thought of as an initial condition, describes the system so that nature knows what the future evolution of the system has to be given that data.

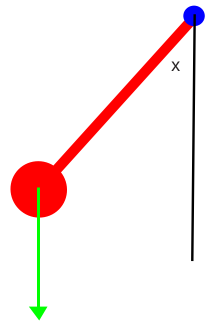


We don't yet know yet the curve $t \mapsto \vec{r}(t) = \langle x(t), y(t)\rangle$, but we know the tangents $\vec{r}\,'(t) = \langle x'(t), y'(t)\rangle = \langle y(t), -x(t)\rangle$. In other words, we know a direction at each point. The equation $(x' = y, y' = -x)$ is called a system of ordinary differential equations (ODE's) More generally, the problem when studying ODE's is to find solutions $x(t), y(t)$ of equations $x'(t) = f(x(t), y(t)), y'(t) = g(x(t), y(t))$. Here we look for curves $x(t), y(t)$ so that at any given point (x, y), the tangent vector $(x'(t), y'(t))$ is $(y, -x)$. You can check by differentiation that the circles $(x(t), y(t)) = (r\sin(t), r\cos(t))$ are solutions. They form a family of curves. Can you interpret these solutions physically?



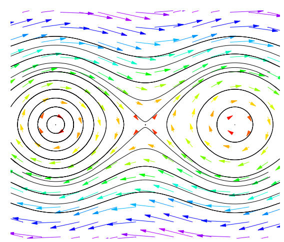
If $x(t)$ is the angle of a pendulum, then the gravity acting on it produces a force $G(x) = -gm\sin(x)$, where m is the mass of the pendulum and where g is a constant. For example, if $x = 0$ (pendulum at bottom) or $x = \pi$ (pendulum at the top), then the force is zero.
The Newton equation "mass times acceleration = force" gives

$$\ddot{x}(t) = -g\sin(x(t)) .$$



The equation of motion for the pendulum $\ddot{x}(t) = -g\sin(x(t))$ can be written with $y = \dot{x}$ also as

$$\frac{d}{dt}(x(t), y(t)) = (y(t), -g\sin(x(t)) .$$



Each possible motion of the pendulum $x(t)$ is described by a curve $\vec{r}(t) = (x(t), y(t))$. Writing down explicit formulas for $(x(t), y(t))$ is in this case not possible with known functions like $\sin, \cos, \exp, \log$ etc. However, one still can understand the curves.

Curves on the top of the picture represent situations where the velocity y is large. They describe the pendulum spinning around fast in the clockwise direction. Curves starting near the point $(0, 0)$, where the pendulum is at a stable rest, describe small oscillations of the pendulum.

Vector fields in weather forecast On weather maps, one can see **isoterms**, curves of constant temperature or **isobars**, curves $p(x, y) = c$ of constant pressure. These are level curves. The wind maps are vector fields. $\vec{F}(x, y)$ is the wind velocity at the point (x, y). The wind velocity $\vec{F}$ is not always normal to the **isobars**, the lines of equal pressure p. The scalar pressure field p and the velocity field $\vec{F}$ depend on time. The equations which describe the weather dynamics are called the **Navier Stokes equations**

$$\frac{d}{dt}\vec{F} + \vec{F} \cdot \nabla\vec{F} = \nu\Delta\vec{F} - \nabla p + f, \mathrm{div}\vec{F} = 0$$

(where we will define the derivatives Δ and div later. This is a **partial differential equation** like the transport equation $u_x - u_y = 0$ we have seen earlier. Finding solutions is not trivial: 1 Million dollars are given to the person proving that the equations have smooth solutions in space.

Section 5.4: Fundamental theorem of line integrals

If $\vec{F}$ is a vector field in the plane or in space and $C : t \mapsto \vec{r}(t)$ is a curve, then $\boxed{\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\ dt}$ is called the **line integral** of $\vec{F}$ along the curve C. The short-hand notation $\int_C \vec{F} \cdot d\vec{r}$ is also used.

In physics, if $\vec{F}(x, y, z)$ is a force field, then the line integral $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt$ is called **work**. In electrodynamics, if $\vec{F}(x, y, z)$ is an electric field, then the line integral $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt$ is called **electric potential**.

Example: let $C : t \mapsto \vec{r}(t) = \langle\cos(t), \sin(t)\rangle$ be a circle parametrized by $t \in [0, 2\pi]$ and let $\vec{F}(x, y) = \langle -y, x\rangle$. Calculate the line integral $I = \int_C \vec{F}(\vec{r}) \cdot d\vec{r}$.

Solution: We have $I = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt = \int_0^{2\pi} (-\cos(t), \sin(t)) \cdot (-\cos(t), \sin(t))\ dt = \int_0^{2\pi} \cos^2(t) + \sin^2(t)\ dt = 2\pi$
Example: Let $\vec{r}(t)$ be a curve given in polar coordinates as $\vec{r}(t) = \cos(t), \phi(t) = t$ defined on $[0, \pi]$. Let $\vec{F}$ be the vector field $\vec{F}(x, y) = (-xy, 0)$. Calculate the line integral $\int_C \vec{F} \cdot d\vec{r}$.

Solution: In Cartesian coordinates, the curve is $r(t) = (\cos^2(t), \cos(t)\sin(t))$. The velocity vector is then $r'(t) = (-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t)) = (x(t), y(t))$. The line integral is

$$\int_0^{\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt = \int_0^{\pi} (\cos^3(t)\sin(t), 0) \cdot (-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t))\ dt$$

$$= -2\int_0^{\pi} \sin^2(t)\cos^4(t)\ dt = -2(t/16 + \sin(2t)/64 - \sin(4t)/64 - \sin(6t)/192)|_0^{\pi} = -\pi/8 .$$

Here is the first generalization of the fundamental theorem of calculus to higher dimensions. It is called the **fundamental theorem of line integrals**.

> If $\vec{F} = \nabla f$, then
>
> $$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\ dt = f(\vec{r}(b)) - f(\vec{r}(a)) .$$

In other words, the line integral is the potential difference between the end points $\vec{r}(b)$ and $\vec{r}(a)$, if $\vec{F}$ is a gradient field.

Example: Let $f(x, y, z)$ be the temperature distribution in a room and let $\vec{r}(t)$ the path of a fly in the room, then $f(\vec{r}(t))$ is the temperature, the fly experiences at the point $\vec{r}(t)$ at time t. The change of temperature for the fly is $\frac{d}{dt}f(\vec{r}(t))$. The line-integral of the temperature gradient ∇f along the path of the fly coincides with the temperature difference between the end point and initial point.

If $\vec{r}(t)$ is parallel to the level curve of f, then $d/dt f(\vec{r}(t)) = 0$ and $\vec{r}\,'(t)$ orthogonal to $\nabla f(\vec{r}(t))$
If $\vec{r}(t)$ is orthogonal to the level curve, then $|d/dt f(\vec{r}(t))| = |\nabla f||\vec{r}\,'(t)|$ and $\vec{r}\,'(t)$ is parallel to $\nabla f(\vec{r}(t))$.

The proof uses the chain rule in the second equality and the fundamental theorem of calculus in the third equality of the following identities:

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\ dt = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)\ dt = \int_a^b \frac{d}{dt} f(\vec{r}(t))\ dt = f(\vec{r}(b)) - f(\vec{r}(a)) .$$

It follows that for a gradient field, the line-integral along any closed curve is zero.
A device which implements a force field which is not a gradient field is called a **perpetual motion machine**. Mathematically, it realizes a force field for which along some closed loops the

energy gain is nonnegative. (By possibly changing the direction, the energy change is positive.) The first law of thermodynamics forbids the existence of such a machine. It is informative to contemplate some of the ideas people have come up with and to see why they don't work.

When is a vector field a gradient field? $\vec{F}(x,y) = \nabla f(x,y)$ implies $P_y(x,y) = Q_x(x,y)$. If this does not hold at some point, $\vec{F} = \langle P, Q \rangle$ is no gradient field. This is called the **component test**. We will see later that the condition $\text{curl}(\vec{F}) = Q_x - P_y = 0$ implies that the field is conservative, if the region satisfies a certain property.

Problem: Let $\vec{F}(x,y) = \langle 2xy^2 + 3x^2, 2yx^2 \rangle$. Find a potential f of $\vec{F} = \langle P, Q \rangle$.

The potential function $f(x,y)$ satisfies $f_x(x,y) = 2xy^2 + 3x^2$ and $f_y(x,y) = 2yx^2$. Integrating the second equation gives $f(x,y) = x^2y^2 + h(x)$. Partial differentiation with respect to x gives $f_x(x,y) = 2xy^2 + h'(x)$ which should be $2xy^2 + 3x^2$ so that we can take $h(x) = x^3$. The potential function is $f(x,y) = x^2y^2 + x^3$.

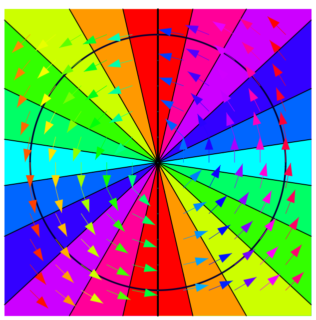
Find g, h from $f(x,y) = \int_0^x P(x,y)\, dx + h(y)$ and $f_y(x,y) = g(x,y)$.

Let $\vec{F}(x,y) = \langle P, Q \rangle = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$. It is a gradient field because $f(x,y) = \arctan(y/x)$ has the property that $f_x = (-y/x^2)/(1 + y^2/x^2) = P$, $f_y = (1/x)/(1 + y^2/x^2) = Q$. However, the line integral $\int_\gamma \vec{F}\, \vec{dr}$, where γ is the unit circle is

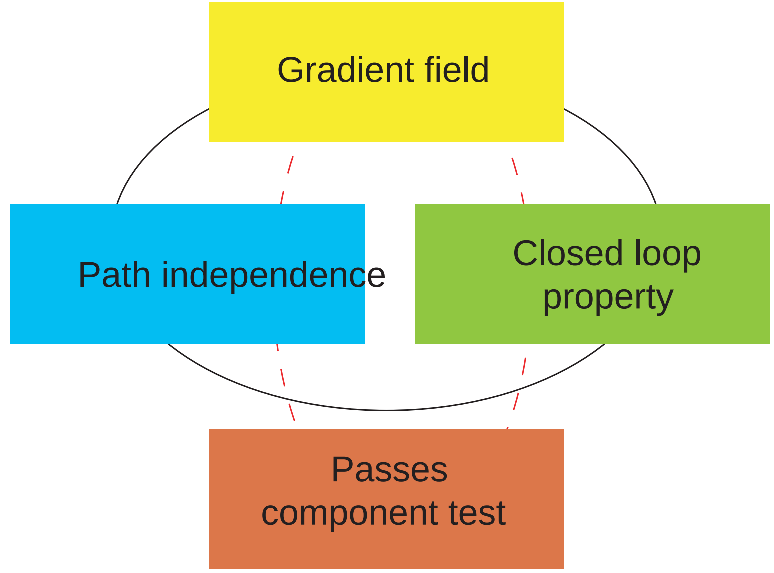
$$\int_0^{2\pi} \langle \frac{-\sin(t)}{\cos^2(t) + \sin^2(t)}, \frac{\cos(t)}{\cos^2(t) + \sin^2} \rangle \cdot \langle -\sin(t), \cos(t) \rangle \, dt$$

which is $\int_0^{2\pi} 1 \, dt = 2\pi$. What is wrong?

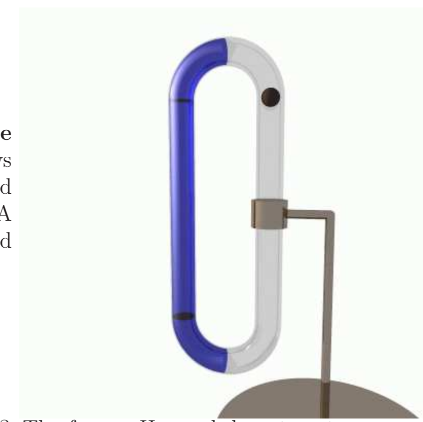
Solution: note that the potential f as well as the vector-field F are not differentiable everywhere. The curl of F is zero except at $(0,0)$, where it is not defined.



We will see later that if R is a simply connected region then $\vec{F}$ is a gradient field if and only if $\text{curl}(F) = 0$ everywhere in R.
A region R is called **simply connected**, if every curve in R can be contracted to a point in a continuous way and every two points can be connected by a path. A disc is an example of a simply connected region, an annular region is an example which is not. Any region with a hole is not simply connected. For simply connected regions, the existence of a gradient field is equivalent to the field having curl zero everywhere.

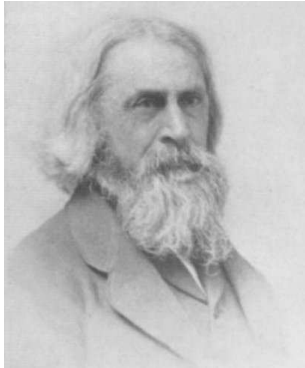


Here is an example of a **pertual motion machine** which implements a nonconservative field and allows so to generate "energy for free". Consider a O-shaped pipe which is filled only on the right side with water. A wooden ball falls on the right hand side in the air and moves up in the water.



Why does this "perpetual motion machine" not work? The former Harvard department professor Benjamin Peirce writes in his book "A system of analytic mechanics" of 1855 and refers so to the "antropic principle":

"Such a series of motions would receive the technical name of a "perpetual motion" by which is to be understood, that of a system which would constantly return to the same position, with an increase of power, unless a portion of the power were drawn off in some way and appropriated, if it were desired, to some species of work. A constitution of the fixed forces, such as that here supposed and in which a perpetual motion would possible, may not, perhaps, be incompatible with the unbounded power of the Creator; but, if it had been introduced into nature, it would have proved destructive to human belief, in the spiritual origin of force, and the necessity of a First Cause superior to matter, and would have subjected the grand plans of Divine benevolence to the will and caprice of man".



Nonconservative fields can also be generated by **optical illusion** as can be seen in **M.C. Escher** pictures. An example is the famous stair in which people always walk down or with the waterfall. The figures suggests the existence of a force field which is not conservative. Can you figure out how Escher's pictures "work"?