Chapter 2. Surfaces and Curves

Section 2.1: Functions, level surfaces, quadrics

A **function of two variables** $f(x, y)$ is usually defined for all points (x, y) in the plane like in the example $f(x, y) = x^2 + \sin(xy)$. In general, we need to restrict the function to a **domain** D in the plane like for $f(x, y) = 1/y$, where (x, y) is defined everywhere except on the x-axes $y = 0$. The **range** of a function f is the set of values which the function f takes. The function $f(x, y) = 1 + x^2$ for example takes all values ≥ 1. The **graph** of $f(x, y)$ is the set $\{(x, y, f(x, y)) \mid (x, y) \in D \}$. The graph of $f(x, y) = \sqrt{x^2 + y^2}$ on the domain $x^2 + y^2 < 1$ is a half sphere. Here are more examples:

example function $f(x, y)$	domain D of f	range $= f(D)$ of f
$f(x, y) = \sin(3x + 3y) - \log(1 - x^2 - y^2)$	open unit disc $x^2 + y^2 < 1$	$[-1, \infty)$
$f(x, y) = f(x, y) = x^2 + y^3 - xy + \cos(xy)$	plane R^2	line
$f(x, y) = \sqrt{4 - x^2 - 2y^2}$	$x^2 + 2y^2 \leq 4$	$[0, 2]$
$f(x, y) = 1/(x^2 + y^2 - 1)$	all except unit circle	all
$f(x, y) = 1/(x^2 + y^2)^2$	all except origin	positive real axis

Given a function $f(x, y)$, the set $f(x, y) = c = $ const is called a **contour curve** or **level curve** of f. For example, for $f(x, y) = 4x^2 + 3y^2$ the level curves $f = c$ are ellipses if $c > 0$. Level curves allow to visualize functions of two variables $f(x, y)$.

Example: For $f(x, y) = x^2 - y^2$. the set $x^2 - y^2 = 0$ is the union of the lines $x = y$ and $x = -y$. The set $x^2 - y^2 = 1$ consists of two hyperbola with with their "noses" at the point $(-1, 0)$ and $(1, 0)$. The set $x^2 - y^2 = -1$ consists of two hyperbola with their noses at $(0, 1)$ and $(0, -1)$.

Drawing several contour curves $\{f(x, y) = c \}$ produces a **contour map**. This allows to visualize a function. For example, the contour curves $\sin(xy) = c$ are the same as the contour curves $xy = C$. As in the previous example, the contour curves of $\sin(xy)$ are hyperbola or a pair of crossing lines.

Contour curves are encountered every day: they appear as **isobars**=curves of constant pressure, or **isoclines**= curves of constant (wind) field direction, They can be **isothermes**= curves of constant temperature or **isoheights** =curves of constant height.
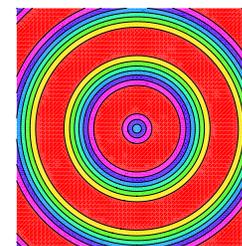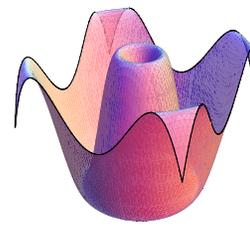
Example: The function $f(x, y) = 1 - 2x^2 - y^2$ has contour curves $f(x, y) = 1 - 2x^2 + y^2 = c$ which are ellipses $2x^2 + y^2 = 1 - c$ for $c < 1$.
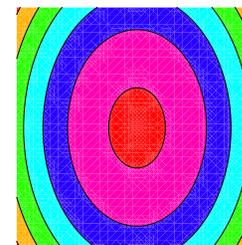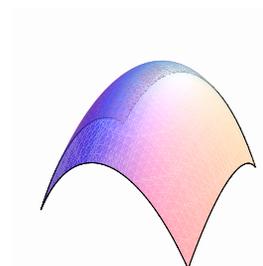
Example: Lets look at the function $f(x, y) = (x^2 - y^2)e^{-x^2 - y^2}$. While we can not find explicit expressions for the contour curves $(x^2 - y^2)e^{-x^2 - y^2} = c$, we can draw the curves with the help of a computer:
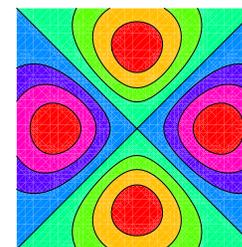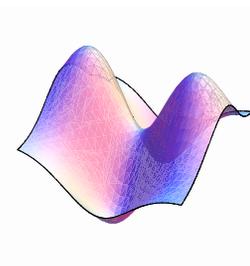
Example: The surface $z = f(x, y) = \sin(\sqrt{x^2 + y^2})$ has concentric circles as contour curves.

In applications, we sometimes have to deal with functions which are not continuous. When plotting the rate of change of temperature of water in relation to pressure and volume for example, one experiences **phase transitions**. Mathematicians have tamed discontinuous events with a mathematical field called "catastrophe theory".

A function $f(x,y)$ is called **continuous** at (a,b) if $f(a,b)$ is finite and $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. This means that for any sequence (x_n,y_n) converging to (a,b) we have $f(x_n,y_n) \to f(a,b)$. Continuity for functions of more than two variables is defined in the same way. Continuity is not always easy to check but fortunately, we most of the time do not have to worry about it. Lets look at some examples:

Example: For $f(x,y) = (xy)/(x^2+y^2)$, we have $\lim_{(x,x)\to(0,0)} f(x,x) = \lim_{x\to 0} x^2/(2x^2) = 1/2$ and $\lim_{(x,0)\to(0,0)} f(0,x) = \lim_{(x,0)\to(0,0)} 0 = 0$. The function is not continuous.

Example: For $f(x,y) = (x^2 y)/(x^2+y^2)$, it is better to describe the function using polar coordinates: $f(r,\theta) = r^3 \cos^2(\theta)\sin(\theta)/r^2 = r\cos^2(\theta)\sin(\theta)$. We see that $f(r,\theta) \to 0$ uniformly if $r \to 0$. The function is continuous.

A function of three variables $g(x,y,z)$ assigns to three variables x,y,z a real number $g(x,y,z)$. The function $g(x,y,z) = 2+\sin(xyz)$ is an example. It could define the temperature distribution in space. We can no more draw a graph of g because that would be an object in 4 dimensions. But we can visualize it differently by drawing **contour surfaces** $g(x,y,z) = c$, where c is constant. For example, for $f(x,y,z) = 4x^2 + 3y^2 + z^2$, the contour surfaces are ellipsoids.

Many surfaces can be described as level surfaces. If this is the case, we call this an **implicit description** of a surface. Here are some examples we know already:

Spheres: The level surfaces of $g(x,y,z) = x^2 + y^2 + z^2$ are spheres.

Graphs: For $g(x,y,z) = z - f(x,y)$ we have the level surface $g = 0$ which is the graph $z = f(x,y)$ of a function of two variables. For example, for $g(x,y,z) = z - x^2 - y^2 = 0$, we have the graph $z = x^2 + y^2$ of the function $f(x,y) = x^2 + y^2$ which is a paraboloid. Note however that most surfaces of the form $g(x,y,z) = c$ can not be written as graphs. The sphere is an example, where we need two graphs to cover it.

Planes: $ax + by + cz = d$ is a plane. With $\vec{n} = \langle a,b,c \rangle$ and $\vec{x} = \langle x,y,z \rangle$, we can rewrite the equation $\vec{n} \cdot \vec{x} = d$. If a point $\vec{x}_0$ is on the plane, then $\vec{n} \cdot \vec{x}_0 = d$. so that $\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0$. This means that every vector $\vec{x} - \vec{x}_0$ in the plane is orthogonal to $\vec{n}$.
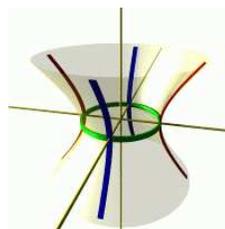
Quadrics: If the function depends only quadratically on variables, that is if $f(x,y,z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + kz + m$ then the surface $f(x,y,z) = 0$ is called a **quadric**. Lets look at a few of them:
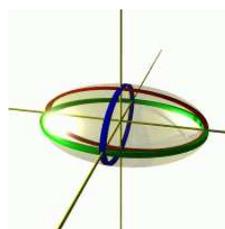
Sphere	Paraboloid	Plane
$(x-a)^2+(y-b)^2+(z-c)^2 = r^2$	$(x-a)^2+(y-b)^2-c = z$	$ax+by+cz = d$
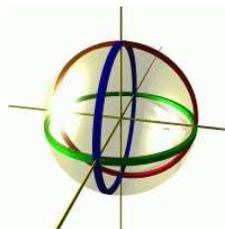
One sheeted Hyperboloid	Cylinder	Two sheeted Hyperboloid
$(x-a)^2+(y-b)^2-(z-c)^2 = r^2$	$(x-a)^2+(y-b)^2 = r^2$	$(x-a)^2+(y-b)^2-(z-c)^2 = -r^2$

Ellipsoid	Hyperbolic paraboloid	Elliptic hyperboloid
$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$	$x^2 - y^2 + z = 1$	$x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$
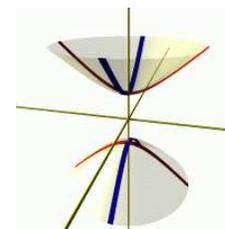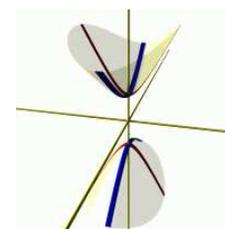
To understand a contour surface, it is very helpful to look at the **traces**, the intersections of the surfaces with the coordinate planes $x = 0, y = 0$ or $z = 0$.

Higher order polynomial surfaces can be intriguingly beautiful. If the function involves only multiplications of variables x, y, z and $x \to f(x, x, x)$ has degree d, then it is called a **degree d polynomial surface**. Degree 2 surfaces are **quadrics**, degree 3 surfaces **cubics**, degree 4 surfaces **quartics**, degree 5 surfaces **quintics**, degree 10 surfaces **decics** and so on.

Section 2.2: Parametrized surfaces

There is a different, fundamentally different way to describe a surface. It is called **parametrization** of a surface. This is achieved with a vector-valued function

$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle.$$

It is given by three functions $x(u, v), y(u, v), z(u, v)$ of two variables. Because two parameters u and v are involved, the map $\vec{r}$ is often called uv-map.
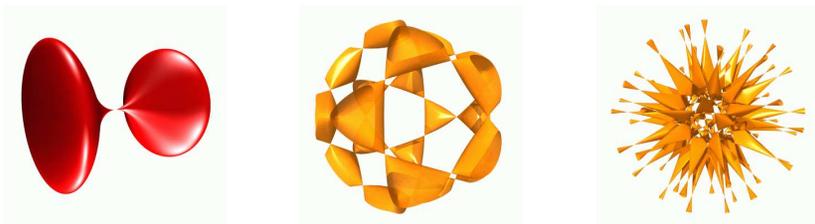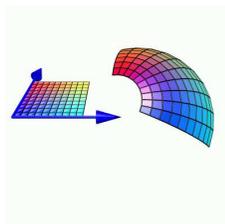

If we keep the first parameter u constant, then $v \mapsto \vec{r}(u, v)$ is a curve on the surface. Similarly, if v is constant, then $u \mapsto \vec{r}(u, v)$ traces a curve the surface. These curves are called **grid curves**. This can be useful to draw the surfaces. We will discuss parametrized curves in the next lecture.

The world of parametric surfaces is fantastic and complex. You will explore this land with the help of the computer algebra system Mathematica. You can survive the parametrization of surfaces topic by keeping in mind 4 important examples. They are *really important* because they: are cases we can understand well and which consequently will return again and again as examples:

1. **Planes**.
Parametric: $\vec{r}(s, t) = \vec{OP} + s\vec{v} + t\vec{w}$

Implicit: $ax + by + cz = d$.

We can change from parametric to implicit using the cross product $\vec{n} = \vec{v} \times \vec{w}$. We can change from implicit to parametric by finding three points P, Q, R on the surface and forming $\vec{u} = \vec{PQ}, \vec{v} = \vec{PR}$.
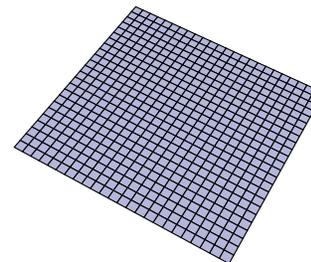
2. **The sphere**:
Parametric: $\vec{r}(u, v) = (a, b, c) + \langle \rho \cos(u) \sin(v), \rho \sin(u) \sin(v), \rho \cos(v) \rangle$.
Implicit: $(x - a)^2 + (y - b)^2 + (z - c)^2 = \rho^2$.

We can go from parametric to implicit by reading off the radius.
Implicit to Parametric: know it
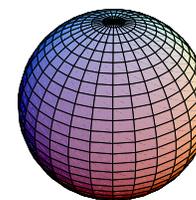
3. **Graphs**:
Parametric: $\vec{r}(u, v) = \langle u, v, f(u, v) \rangle$
Implicit: $z - f(x, y) = 0$.

Parametric to Implicit: think about $z = f(x, y)$
Implicit to Parametric: use x and y as the parameterizations.
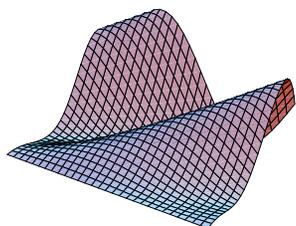
4. **Surfaces of revolution**:
Parametric: $r(u, v) = (g(v) \cos(u), g(v) \sin(u), v)$
Implicit: $\sqrt{x^2 + y^2} = r = g(z)$ can be written as $x^2 + y^2 = g(z)^2$.

Parametric to Implicit: read off the function $g(z)$ the distance to the z-axis.
Implicit to Parametric: use the function g.

A point (x, y) in the plane has the **polar coordinates** $r = \sqrt{x^2 + y^2}, \theta = \text{arctg}(y/x)$. We have $(x, y) = (r \cos(\theta), r \sin(\theta))$. Note that $\theta = \text{arctg}(y/x)$ defines the angle θ only up to an addition of π. The points (x, y) and $(-x, -y)$ would have the same θ. In order to get the correct θ, one can take $\arctan(y/x)$ in $(-\pi/2, \pi/2]$, where $\pi/2$ is the value when $y/x = \infty$, and add π if $x < 0$ or $x = 0, y < 0$. If we represent points as
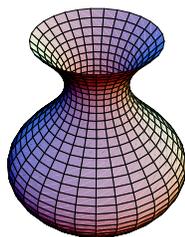
$$(x, y, z) = (r \cos(\theta), r \sin(\theta), z)$$

we speak of **cylindrical coordinates**. Here are some level surfaces in cylindrical coordinates: $r = 1$ is a **cylinder**, $r = |z|$ is a **double cone**, $\theta = 0$ is a **half plane**, $r = \theta$ is a **rolled sheet of paper** and and $r = 2 + \sin(z)$ is an example of a **surface of revolution**.

Spherical coordinates use the distance ρ to the origin as well as two angles θ and ϕ. The first angle θ is the polar angle as in polar coordinates. The angle ϕ is the angle between the vector $\vec{OP}$ and the z-axis. A point has the **spherical coordinate**

$$(x, y, z) = (\rho \cos(\theta) \sin(\phi), \rho \sin(\theta) \sin(\phi), \rho \cos(\phi)) .$$

There are two important pictures. The distance to the z axes $r = \rho \sin(\phi)$ and the height $z = \rho \cos(\phi)$ can be read off by the left picture, the coordinates $x = r \cos(\theta), y = r \sin(\theta)$ from the right picture.

$$x = \rho \cos(\theta) \sin(\phi),$$
$$y = \rho \sin(\theta) \sin(\phi),$$
$$z = \rho \cos(\phi)$$
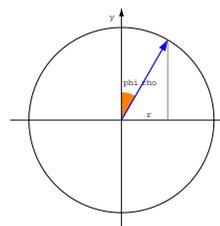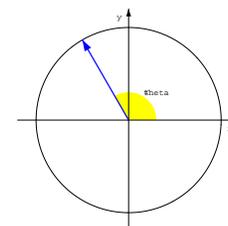
Here are some level surfaces described in spherical coordinates: $\rho = 1$ is a **sphere**, the surface $\phi = \pi/2$ is a **single cone**, $\rho = \phi$ is an **apple shaped surface** and $\rho = 2 + \cos(3\theta) \sin(\phi)$ is an example of a **bumpy sphere**.

Section 2.3: Parametrized curves

If $x(t), y(t)$ are two functions of a variable t, which is defined in a **parameter interval** $I = [a, b]$, then $\vec{r}(t) = \langle f(t), g(t) \rangle$ is called a **parametric curve** in the plane. The functions $x(t), y(t)$ are called **coordinate functions**. For example, $\vec{r}(t) = \langle \cos(3t), \sin(5t) \rangle$ describes such a curve.

In three dimensions, we describe curves with three functions $x(t), y(t), z(t)$. The parametrization is $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$. The image of r is a **space curve**. We always think of the **parameter** t as **time**. For a fixed time t, we have a vector $\langle x(t), y(t), z(t) \rangle$ in space. As t varies, the end point of this vector moves along the curve. The parametrization contains more information about the curve then the curve alone. It tells also, how fast we go along the curve.

Example: If $x(t) = t$, $y(t) = f(t)$, the curve $\vec{r}(t) = \langle t, f(t) \rangle$ traces the graph of $f(x)$. For example, for $f(x) = x^2 + 1$, the curve is a parabola.

Example: With $x(t) = \cos(t), y(t) = \sin(t)$, then $\vec{r}(t)$ follows a **circle**. We can see this from $x(t)^2 + y(t)^2 = 1$.

Example: With $x(t) = \cos(t), y(t) = \sin(t), z(t) = t$ we get a **space curve** $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$. It traces a **helix**.

Example: If $x(t) = \cos(2t), y(t) = \sin(2t), z(t) = 2t$, then we have the same curve as in the previous example but the curve is traversed **faster**. The **parameterization** of the curve has changed.

Example: If $x(t) = \cos(-t), y(t) = \sin(-t), z(t) = -t$, then we have the same curve again but we traverse it in the **opposite direction**.

Example: If $P = (a, b, c)$ and $Q = (u, v, w)$ are points in space, then $\vec{r}(t) = \langle a + t(u - a), b + t(v - b), c + t(w - c) \rangle$ defined on $t \in [0, 1]$ is a **line segment** connecting P with Q. For example,

$\vec{r}(t) = \langle 1+t, 1-t, 2+3t \rangle$ connects the points $P = (1,1,2)$ with $Q = (2,0,1)$.

Sometimes it is possible to eliminate the parameter t and write the curve using equations. We need one equation in the plane or two equations in space. We have seen this already with lines.

Example: for $x(t) = t\cos(t), y(t) = t\sin(t), z(t) = t$, then $x = t\cos(z), y = t\sin(z)$ and we can see that $x^2 + y^2 = z^2$. The curve is located on a cone. We also have $x/y = \tan(z)$ so that we could see the curve as an intersection of two surfaces. While seeing a surface on which the curve is can help to understand the surface, seeing it as an intersection of two surfaces is not so useful.

Curves describe the paths of particles, celestial bodies, or quantities which change in time. Examples are the motion of a star moving in a galaxy, or economical data changing in time.. Here are some more places, where curves appear:

Strings or knots	are closed curves in space.
Large Molecules	like RNA or proteins can be modeled as curves.
Computer graphics:	surfaces are represented by mesh of curves.
Typography:	fonts represented by Bezier curves.
Space time:	curve in space-time describes the motion of an object
Topology:	space filling curves, boundaries of surfaces or knots.

If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a curve, then $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle = \langle \dot{x}, \dot{y}, \dot{z} \rangle$ is called the **velocity** at time t. Its length $|\vec{r}'(t)|$ is called **speed** and $\vec{v}/|\vec{v}|$ is called **direction of motion**. The vector $\vec{r}\,''(t)$ is called the **acceleration**. The third derivative $\vec{r}\,'''$ is called the **jerk**. We have:

The velocity vector $\vec{r}\,'(t)$ is tangent to the curve at $\vec{r}(t)$.

Here is an example where velocities, acceleration and jerk are computed:

Position	$\vec{r}\,(t)$	$= \langle \cos(3t), \sin(2t), 2\sin(t) \rangle$
Velocity	$\vec{r}\,'(t)$	$= \langle -3\sin(3t), 2\cos(2t), 2\cos(t) \rangle$
Acceleration	$\vec{r}\,''(t)$	$= \langle -9\cos(3t), -4\sin(2t), -2\sin(t) \rangle$
Jerk	$\vec{r}\,'''(t)$	$= \langle 27\sin(3t), 8\cos(2t), -2\cos(t) \rangle$

Lets look at some examples of velocities and accelerations:

| Signals in nerves:	40 m/s	Train:	0.1-0.3 m/s^2
Plane:	70-900 m/s	Car:	3-8 m/s^2
Sound in air:	Mach1=340 m/s	Free fall:	1G = 9.81 m/s^2
Speed of bullet:	1200-1500 m/s	Space shuttle:	3G = 30m/s^2
Earth around the sun:	30'000 m/s	Combat plane F16:	9G m/s^2
Sun around galaxy center:	200'000 m/s	Ejection from F16:	14G m/s^2.
Light in vacuum:	300'000'000 m/s	Electron in vacuum tube:	10^{15} m/s^2

The **differentiation rules** in one dimension $(f+g)' = f' + g'$ (addition rule) $(cf)' = cf'$, $(fg)' = f'g + fg'$ (Leibniz), $(f(g))' = f'(g)g'$ (chain rule) generalize to vector-valued functions:

$(\vec{v} + \vec{w})' = \vec{v}' + \vec{w}'$, $(c\vec{v})' = c\vec{v}'$, $(\vec{v} \cdot \vec{w})' = \vec{v}' \cdot \vec{w} + \vec{v} \cdot \vec{w}'$ $(\vec{v} \times \vec{w})' = \vec{v}' \times \vec{w} + \vec{v} \times \vec{w}'$ (Leibniz), $(\vec{v}(f(t)))' = \vec{v}'(f(t))f'(t)$ (chain rule).

The process of differentiation of a curve can be reversed. If $\vec{r}'(t)$ and $\vec{r}(0)$ is known, we can figure out $\vec{r}(t)$ by **integration** $\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{r}'(s)\,ds$.

Assume we know the acceleration $\vec{a}(t) = \vec{r}''(t)$ as well as initial velocity and position $\vec{r}'(0)$ and $\vec{r}(0)$. Then $\vec{r}(t) = \vec{r}(0) + t\vec{r}'(0) + \vec{R}(t)$, where $\vec{R}(t) = \int_0^t \vec{v}(s)\,ds$ and $\vec{v}(t) = \int_0^t \vec{a}(s)\,ds$.

Lets look the important example of **free fall**, which is the case when acceleration is constant. If $\vec{r}''(t) = \langle 0, 0, -10 \rangle, \vec{r}'(0) = \langle 0, 1000, 2 \rangle, \vec{r}(0) = \langle 0, 0, h \rangle$, then $\vec{r}(t) = \langle 0, 1000t, h + 2t - 10t^2/2 \rangle$.
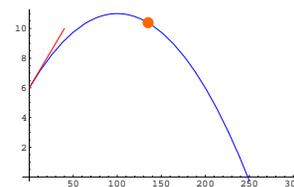
Section 2.4: Arc length and curvature

If $t \in [a, b] \mapsto \vec{r}(t)$ with velocity $\vec{r}\,'(t)$ and speed $|\vec{r}\,'(t)|$, then

$$L = \int_a^b |\vec{r}\,'(t)|\,dt$$

is called the **arc length of the curve**. It can be written out. For space curves for example we have

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$$

Example: The arc length of the circle $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$ parameterized by $0 \le t \le 2\pi$ is $2pi$ because the speed $|\vec{r}'(t)|$ is constant and equal to 1.

Example: The helix $\vec{r}(t) = (\cos(t), \sin(t), t)$ has velocity $\vec{r}\,'(t) = (-\sin(t), \cos(t), 1)$ and constant speed $|\vec{r}\,'(t)| = (-\sin(t), \cos(t), 1) = \sqrt{2}$.

Example: Lets compute the arc length of the curve

$$\vec{r}(t) = \langle t, \log(t), t^2/2 \rangle .$$

for $1 \le t \le 2$. Because $\vec{r}'(t) = \langle 1, 1/t, t \rangle$, we have $\vec{r}'(t) = \sqrt{1 + \frac{1}{t^2} + t^2} = |\frac{1}{t} + t|$. We have $L = \int_1^2 \frac{1}{t} + t\,dt = \log(t) + \frac{t^2}{2}|_1^2 = \log(2) + 2 - 1/2$. This curve does not have a name. But because it is constructed in such a way that the arc length can be computed, we an call it "opportunity curve".

Example: What is the arc length of the curve $\vec{r}(t) = \langle \cos^3(t), \sin^3(t) \rangle$? We have $|\vec{r}'(t)| = 3\sqrt{\sin^2(t)\cos^4(t) + \cos^2(t)\sin^4(t)} = (3/2)|\sin(2t)|$. Therefore, $\int_0^{2\pi}(3/2)\sin(2t)\,dt = 6$.

Example: Find the arc length of $\vec{r}(t) = \langle t^2/2, t^3/3 \rangle$ for $-1 \le t \le 1$. This cubic curve satisfies $y^2 = x^3 8/9$ and is an example of an **elliptic curve**. Because $\int x\sqrt{1+x^2}\ dx = (1+x^2)^{3/2}/3$, the integral can be evaluated as $\int_{-1}^{1} |x|\sqrt{1+x^2}\ dx = 2\int_0^1 x\sqrt{1+x^2}\ dx = 2(1+x^2)^{3/2}/3|_0^1 = 2(2\sqrt{2}-1)/3$.

Example: The arc length of an **epicycle** $\vec{r}(t) = \langle t+\sin(t), \cos(t) \rangle$ parameterized by $0 \le t \le 2\pi$. We have $|\vec{r'}(t)| = \sqrt{2+2\cos(t)}$. so that $L = \int_0^{2\pi} \sqrt{2+2\cos(t)}\ dt$. A **substitution** $t = 2u$ gives $L = \int_0^\pi \sqrt{2+2\cos(2u)}\ 2du = \int_0^\pi \sqrt{2+2\cos^2(u)-2\sin^2(u)}\ 2du = \int_0^\pi \sqrt{4\cos^2(u)}\ 2du = 4\int_0^\pi |\cos(u)|\ du = 8$.

Example: the arc length of the **catenary** $\vec{r}(t) = \langle t, \cosh(t) \rangle$, where $\cosh(t) = (e^t + e^{-t})/2$ is the **hyperbolic cosine** and $t \in [-1, 1]$. We have

$$\cosh^2(t)^2 - \sinh^2(t) = 1 \ ,$$

where $\sinh(t) = (e^t - e^{-t})/2$ is the **hyperbolic sine**.

Because a parameter change $t = t(s)$ corresponds to a **substitution** in the integration which does not change the integral, we immediately have

> The arc length is independent of the parameterization of the curve.

Example: the circle parameterized by $\vec{r}(t) = \langle \cos(t^2), \sin(t^2) \rangle$ on $t = [0, \sqrt{2\pi}]$ has the velocity $\vec{r'}(t) = 2t(-\sin(t), \cos(t))$ and speed $2t$. The arc length is still $\int_0^{\sqrt{2\pi}} 2t\ dt = t^2|_0^{\sqrt{2\pi}} = 2\pi$.

Often, there is no closed formula for the arc length of a curve. For example, the **Lissajous figure** $\vec{r}(t) = \langle \cos(3t), \sin(5t) \rangle$ leads to the arc length integral $\int_0^{2\pi} \sqrt{9\sin^2(3t) + 25\cos^2(5t)}\ dt$ which can only be evaluated numerically.

Define the **unit tangent vector** $\vec{T}(t) = \vec{r'}(t)|/|\vec{r'}(t)|$ **unit tangent vector**. The **curvature** if a curve at the point $\vec{r}(t)$ is defined as

$$\kappa(t) = \frac{|\vec{T'}(t)|}{|\vec{r'}(t)|}$$

The curvature is the length of the acceleration vector if $\vec{r}(t)$ traces the curve with constant speed 1. A large curvature at a point means that the curve is strongly bent. Unlike the acceleration or the velocity, the curvature does not depend on the parameterization of the curve. You "see" the curvature, while you "feel" the acceleration.

Example: The curve $\vec{r}(t) = \langle t, f(t) \rangle$, which is the graph of a function f has the velocity $\vec{r'}(t) = (1, f'(t))$ and the unit tangent vector $\vec{T}(t) = (1, f'(t))/\sqrt{1+f'(t)^2}$. After some simplification we get

$$\kappa(t) = |\vec{T'}(t)|/|\vec{r'}(t)| = |f''(t)|/\sqrt{1+f'(t)^2}^3$$

For example, for $f(t) = \sin(t)$, then $\kappa(t) = |\sin(t)|/\sqrt{1+\cos^2(t)}^3$.

If $\vec{r}(t)$ is a curve which has nonzero speed at t, then we can define

$\vec{T}(t) = \frac{\vec{r'}(t)}{|\vec{r'}(t)|}$, the **unit tangent vector**

$\vec{N}(t) = \frac{\vec{T'}(t)}{|\vec{T'}(t)|}$, the **normal vector** and

$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ the **bi-normal vector**

If we differentiate $\vec{T}(t) \cdot \vec{T}(t) = 1$, we get $\vec{T'}(t) \cdot \vec{T}(t) = 0$ and see that $\vec{N}(t)$ is perpendicular to $\vec{T}(t)$. The three vectors $(\vec{T}(t), \vec{N}(t), \vec{B}(t))$ are unit vectors orthogonal to each other.

Here is an application of curvature: If a curve $\vec{r}(t)$ represents a **wave front** and $\vec{n}(t)$ is a **unit vector normal** to the curve at $\vec{r}(t)$, then $\vec{s}(t) = \vec{r}(t) + \vec{n}(t)/\kappa(t)$ defines a new curve called the **caustic** of the curve. Geometers call that curve the **evolute** of the original curve.
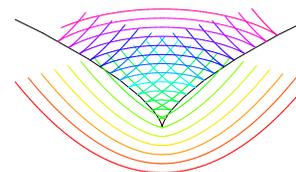
A useful formula for curvature is

$$\kappa(t) = \frac{|\vec{r'}(t) \times \vec{r''}(t)|}{|\vec{r'}(t)|^3}$$

which we prove in class. Finally, lets mention that curvature is important also in **computer vision**. If the gray level value of a picture is modeled as a function $f(x, y)$ of two variables, places where the level curves of f have maximal curvature corresponds to **corners** in the picture. This is useful when **tracking** or **identifying** objects.
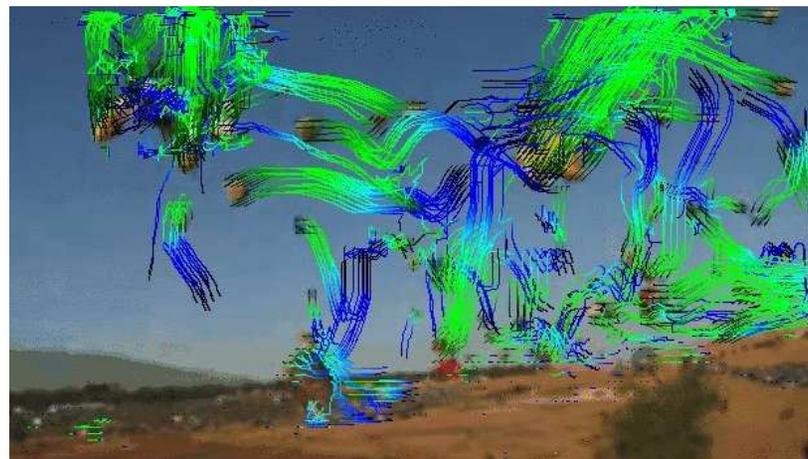
Tracking balloons in a movie taken at a balloon festival in Albuquerque. The program computes curvature in order to identify interesting points, then tracks them over time.