

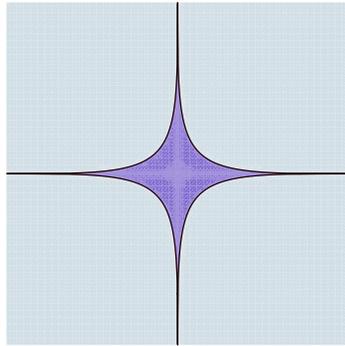
## Homework for Chapter 6. Integral Theorems

### Section 6.1:

- (Greens theorem) Calculate the line integral  $\int_C \vec{F} \cdot d\vec{r}$  with  $\vec{F} = \langle 2y + x \sin(y), x^2 \cos(y) - 3y^{200 \sin(y)} \rangle$  along a triangle  $C$  with edges  $(0, 0)$ ,  $(\pi/2, 0)$  and  $(\pi/2, \pi/2)$ .
- (Greens theorem) Evaluate the line integral of the vector field  $\vec{F}(x, y) = \langle xy^2, x^2 \rangle$  along the rectangle with vertices  $(0, 0)$ ,  $(2, 0)$ ,  $(2, 3)$ ,  $(0, 3)$ .
- (Greens theorem) Find the area of the region bounded by the **hypocycloid**

$$\vec{r}(t) = \langle \cos^3(t), \sin^3(t) \rangle$$

using Green's theorem. The curve is parameterized by  $t \in [0, 2\pi]$ .



- Let  $G$  be the region  $x^6 + y^6 \leq 1$ . Compute the line integral of the vector field  $\vec{F}(x, y) = \langle x^6, y^6 \rangle$  along the boundary.
- Let  $\vec{F}(x, y) = \langle -y/(x^2 + y^2), x/(x^2 + y^2) \rangle$ . Let  $C : \vec{r}(t) = \langle \cos(t), \sin(t) \rangle, t \in [0, 2\pi]$ .
  - Compute  $\int_C \vec{F} \cdot d\vec{r}$ .
  - Show that  $\text{curl}(\vec{F}) = 0$  everywhere for  $(x, y) \neq (0, 0)$ .
  - Let  $f(x, y) = \arctan(y/x)$ . Verify that  $\nabla f = \vec{F}$ .
  - Why do a) and b) not contradict the fact that a gradient field has the closed loop property? Why does a) and b) not contradict Green's theorem?

### Section 6.2: Curl, Div and Flux

- (Curl and Div) Find a nonzero vector field  $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$  in each of the following cases:
  - $\vec{F}$  is irrotational but not incompressible.
  - $\vec{F}$  is incompressible but not irrotational.
  - $\vec{F}$  is irrotational and incompressible.
  - $\vec{F}$  is not irrotational and not incompressible.



The terminology in this problem comes from fluid dynamics where fluids can be incompressible, irrotational.

- (Curl) The vector field  $\vec{F}(x, y, z) = \langle x, y, -2z \rangle$  satisfies  $\text{div}(\vec{F}) = 0$ . Can you find a vector field  $\vec{G}(x, y, z)$  such that  $\text{curl}(\vec{G}) = \vec{F}$ ? Such a field  $\vec{G}$  is called a **vector potential**.  
**Hint.** Write  $\vec{F}$  as a sum  $\langle x, 0, -z \rangle + \langle 0, y, -z \rangle$  and find vector potentials for each of the summand using a vector field you have seen in class.
- (Flux integral) Evaluate the flux integral  $\int_S \langle 0, 0, yz \rangle \cdot d\vec{S}$ , where  $S$  is the surface with parametric equation  $x = uv, y = u + v, z = u - v$  on  $R : u^2 + v^2 \leq 1$ .
- (Flux integral) Evaluate the flux integral  $\iint_S \text{curl}(F) \cdot d\vec{S}$  for  $\vec{F}(x, y, z) = \langle xy, yz, zx \rangle$ , where  $S$  is the part of the paraboloid  $z = 4 - x^2 - y^2$  that lies above the square  $[1, 0] \times [0, 1]$  and has an upward orientation.
- (Flux integral) a) What is the relation between the flux of the vector field  $\vec{F} = \nabla g / |\nabla g|$  through the surface  $S : \{g = 1\}$  with  $g(x, y, z) = x^6 + y^4 + 2z^8$  and the surface area of  $S$ ?  
b) Find the flux of the vector field  $\vec{G} = \nabla g \times \langle 0, 0, 1 \rangle$  through the surface  $S$ .

**Remark** This problem, both part a) and part do not need any computation. You can answer each question with one sentence. In part a) compare  $\vec{F} \cdot d\vec{S}$  with  $dS$  in that case.

### Section 6.3: Stokes theorem

- 1) (Stokes theorem) Find  $\int_C \vec{F} \cdot d\vec{r}$ , where  $\vec{F}(x, y, z) = \langle x^2y, x^3/3, xy \rangle$  and  $C$  is the curve of intersection of the hyperbolic paraboloid  $z = y^2 - x^2$  and the cylinder  $x^2 + y^2 = 1$ , oriented counterclockwise as viewed from above.
- 2) (Stokes theorem) If  $S$  is the surface  $x^6 + y^6 + z^6 = 1$  and assume  $\vec{F}$  is a smooth vector field. Explain why  $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = 0$ .
- 3) (Stokes theorem) Evaluate the flux integral

$$\iint_S \text{curl}(\vec{F}) \cdot d\vec{S},$$

where  $\vec{F}(x, y, z) = \langle xe^{yz}z^3 + 2xyze^{x^2+z}, x + z^2e^{x^2+z}, ye^{x^2+z} + ze^x \rangle$  and where  $S$  is the part of the ellipsoid  $x^2 + y^2/4 + (z+1)^2 = 2$ ,  $z > 0$  oriented so that the normal vector points upwards.

- 4) (Stokes theorem) Find the line integral  $\int_C \vec{F} \cdot d\vec{r}$ , where  $C$  is the circle of radius 3 in the  $xz$ -plane oriented counter clockwise when looking from the point  $(0, 1, 0)$  onto the plane and where  $\vec{F}$  is the vector field

$$\vec{F}(x, y, z) = \langle 2x^2z + x^5, \cos(e^y), -2xz^2 + \sin(\sin(z)) \rangle.$$

**Hint.** Use a convenient surface  $S$  which has  $C$  as a boundary.

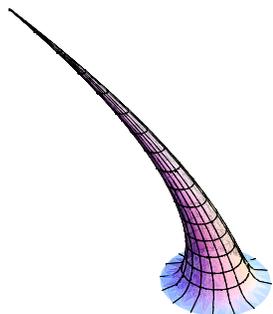
- 5) (Stokes theorem) Find the flux integral  $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$ , where

$$\vec{F}(x, y, z) = \langle 2 \cos(\pi y)e^{2x} + z^2, x^2 \cos(z\pi/2) - \pi \sin(\pi y)e^{2x}, 2xz \rangle$$

and  $S$  is the **thorn** surface parametrized by

$$\vec{r}(s, t) = \langle (1 - s^{1/3}) \cos(t) - 4s^2, (1 - s^{1/3}) \sin(t), 5s \rangle$$

with  $0 \leq t \leq 2\pi, 0 \leq s \leq 1$  and oriented so that the normal vectors point to the outside of the thorn.

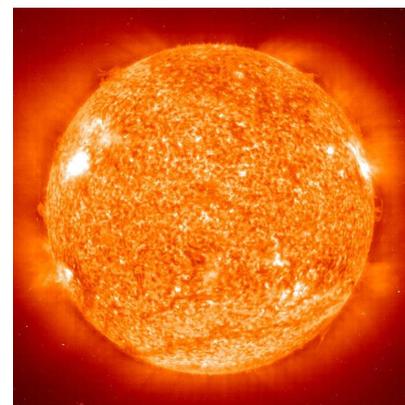


### Section 6.4:

- 1) (Divergence theorem) Compute using the divergence theorem the flux of the vector field  $\vec{F}(x, y, z) = \langle 3y, xy, 2yz \rangle$  through the unit cube  $[0, 1] \times [0, 1] \times [0, 1]$ .
- 2) (Divergence theorem) Find the flux of the vector field  $\vec{F}(x, y, z) = \langle xy, yz, zx \rangle$  through the solid cylinder  $x^2 + y^2 \leq 1, 0 \leq z \leq 1$ .
- 3) (Divergence theorem) Use the divergence theorem to calculate the flux of  $\vec{F}(x, y, z) = \langle x^3, y^3, z^3 \rangle$  through the sphere  $S : x^2 + y^2 + z^2 = 1$  where the sphere is oriented so that the normal vector points outwards.
- 4) (Divergence theorem) Assume the vector field

$$\vec{F}(x, y, z) = \langle 5x^3 + 12xy^2, y^3 + e^y \sin(z), 5z^3 + e^y \cos(z) \rangle$$

is the magnetic field of the **sun** whose surface is a sphere of radius 3 oriented with the outward orientation. Compute the magnetic flux  $\iint_S \vec{F} \cdot d\vec{S}$ .



- 5) (Divergence theorem) Find  $\int \int_S \vec{F} \cdot d\vec{S}$ , where  $\vec{F}(x, y, z) = \langle x, y, z \rangle$  and  $S$  is the outwardly oriented surface obtained by removing the cube  $[1, 2] \times [1, 2] \times [1, 2]$  from the cube  $[0, 2] \times [0, 2] \times [0, 2]$ .

