Oliver Knill, Harvard Summer School, Summer 2010

Homework for Chapter 5. Triple integrals and line integrals

Section 5.1: Triple integrals

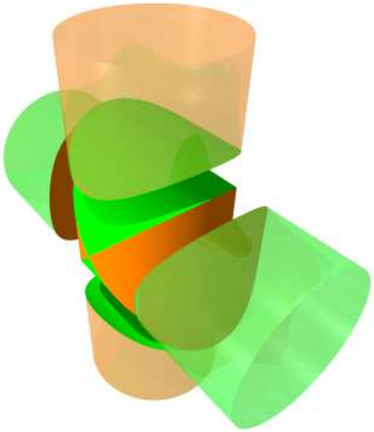
1) (triple integrals) Evaluate the triple integral
$$\int_0^1 \int_0^z \int_0^{2y} z e^{-y^2} \, dxdydz \, .$$

2) (triple integrals polar coordinates) Find the volume of the solid bounded by the paraboloids $z = x^2 + y^2$ and $z = 9 - (x^2 + y^2)$ and satisfying $x \geq 0$.

3) (triple integrals, polar coordinates) Find the **moment of inertia** $\int\int\int_E (x^2 + y^2) \, dV$ of a cone
$$E = \{ x^2 + y^2 \leq z^2 \; 0 \leq z \leq 1 \} \, ,$$
which has the z-axis as its center of symmetry.

4) (triple integrals) Integrate $f(x, y, z) = x^2 + y^2 - z$ over the tetrahedron with vertices $(0, 0, 0), (1, 1, 0), (0, 1, 0), (0, 0, 3)$.

5) (triple integral) This is an old classic problem: What is the volume of the body obtained by intersecting the solid cylinders $x^2 + z^2 \leq 1$ and $y^2 + z^2 \leq 1$?



Section 5.2: Spherical and cylindrical coordinates

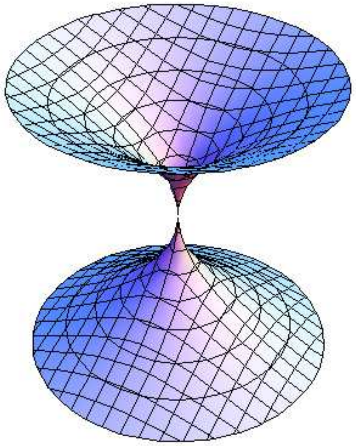
1) (cylindrical or spherical coordinates?) The density of a solid $E = x^2 + y^2 - z^2 < 1, -1 < z < 1$. is given by the forth power of the distance to the z-axes: $\sigma(x, y, z) = (x^2 + y^2)^2$. Find its mass
$$M = \int\int\int_E (x^2 + y^2)^2 \, dxdydz \, .$$

2) (cylindrical or spherical coordinates?) Find the moment of inertia $\int\int\int_E (x^2 + y^2) \, dV$ of the body E whose volume is given by the integral
$$\mathrm{Vol}(E) = \int_0^{\pi/4} \int_0^{\pi/2} \int_0^3 \rho^2 \sin(\phi) \, d\rho d\theta d\phi \, .$$

3) (integration in spherical coordinates?) A solid is described in spherical coordinates by the inequality $\rho \leq \sin(\phi)$. Find its volume.

4) (cylindrical or spherical coordinates?) Integrate the function
$$f(x, y, z) = e^{(x^2+y^2+z^2)^{3/2}}$$
over the solid which lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$, which is in the first octant and which is above the cone $x^2 + y^2 = z^2$.

5) (cylindrical or spherical coordinates?) Find the volume of the solid $x^2 + y^2 \leq z^4, z^2 \leq 1$.



Section 5.3: Vector fields and line integrals

1) (vector fields) The vector field $\vec{F}(x,y) = \langle x/r^3, y/r^3 \rangle$ appears in electrostatics, where $r = \sqrt{x^2+y^2}$ is the distance to the charge. Find a function $f(x,y)$ such that $\vec{F} = \nabla f$.

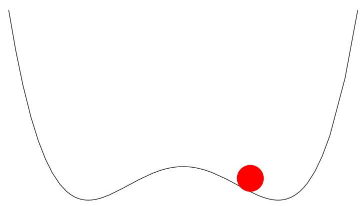
   Hint. Write out the vector field so that each component P, Q is an explicit function in x, y. Then integrate P with respect to x.

2) (vector fields) a) Draw the gradient vector field of the function $f(x,y) = \sin(x+y)$.
   b) Draw the gradient vector field of the function $f(x,y) = (x-1)^2 + (y-2)^2$.
   Hint: In both cases, draw first a contour map of f and use a property of gradients to draw the vector field $F(x,y) = \nabla f$.

3) (vector fields)
   a) Is the vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle = \langle xy, x^2 \rangle$ a gradient field?
   b) Is the vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle = \langle \sin(x) + y, \cos(y) + x \rangle$ a gradient field?
   In both cases, give the potential if it exists and if there is no gradient, give a reason, why it is not a gradient field.

4) (vector fields) A ball rolls on the graph of the function $f(x) = x^4 - x^2$. Its position at time t is $x(t)$. The ball feels the acceleration $x''(t) = -f'(x)$. (The sign is chosen as given because for a positive slope $f'(x) > 0$ the ball feels a negative acceleration and for a negative slope $f'(x) < 0$, the ball is accelerated.) If the motion of the ball is described in coordinates $(x(t), y(t))$, where $y(t) = x'(t)$ is the velocity of the ball, the corresponding vector field is $\vec{F}(x,y) = \langle y, -f'(x) \rangle$. Because $\vec{r}\,'(t) = \langle x'(t), y'(t) \rangle = \langle y(t), -f'(x(t)) \rangle$, the flow lines $\vec{r}(t) = \langle x(t), y(t) \rangle$ describe the position and velocity of the ball at time t. Draw the vector field $F(x,y)$, draw a few typical flow lines in the plane and match these curves with the corresponding motion of the ball.

   Hint. It helps to look at the places, where the vector field is zero. These are called "equilibrium points" and correspond to situations, where the ball does not move.



5) (vector field) The vector field

$$\vec{F}(x,y,z) = \langle 5x^4y + z^4 + y * \cos(x*y), x^5 + x * \cos(x*y), 4xz^3 \rangle$$

is a gradient field. Find the potential function f.

Section 5.4: Fundamental theorem of line integrals

1) (line integrals) Let C be the space curve $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ for $t \in [0,1]$ and let $\vec{F}(x,y,z) = \langle y, x, 5 \rangle$. Calculate the line integral $\int_C \vec{F} \cdot \vec{dr}$.

2) (line integrals) Find the work done by the force field $F(x,y) = (x\sin(y), y)$ on a particle that moves along the parabola $y = x^2$ from $(-1,1)$ to $(2,4)$.

3) (line integrals) Let $\vec{F}$ be the vector field $\vec{F}(x,y) = \langle -y, x \rangle/2$. Compute the line integral of F along an ellipse $\vec{r}(t) = \langle a\cos(t), b\sin(t) \rangle$ with width $2a$ and height $2b$. The result should depend on a and b.

4) (line integrals) After this summer school, you relax in a Jacuzzi and move along curve C which is given by part of the curve $x^{10} + y^{10} = 1$ in the first quadrant, oriented counter clockwise. The hot water in the tub has the velocity $\vec{F}(x,y) = \langle x, y^4 \rangle$. Calculate the line integral $\int_C \vec{F} \cdot \vec{dr}$, the energy you gain from the fluid force.

5) (line integrals) Find a closed curve $C : \vec{r}(t)$ for which the vector field

$$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle = \langle xy, x^2 \rangle$$

satisfies $\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt \neq 0$.