Oliver Knill, Harvard Summer School, Summer 2010

# Homework for Chapter 4. Extrema and Double integrals

## Section 4.1: Extrema

1) (extrema) Find all the extrema of the function $f(x,y) = 2x^3 + 4y^2 - 2y^4 - 6x$ and determine whether they are maxima, minima or saddle points.

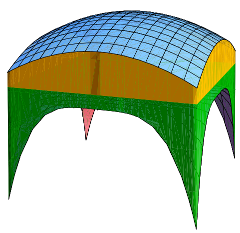
2) (extrema) Where on the parametrized surface $\vec{r}(u,v) = \langle u^2, v^3, uv \rangle$ is the temperature $T(x,y,z) = 12x + y - 12z$ minimal? To find the minimum, look where the function $f(u,v) = T(\vec{r}(u,v))$ has an extremum. Find all local maxima, local minima or saddle points of $f$.

   **Remark.** After you have found the function $f(u,v)$, you could replace the variables $u, v$ again with $x, y$ if you like and look at a function $f(x,y)$.

3) (extrema) Find and classify all the extrema of the function $f(x,y) = e^{-x^2-y^2}(x^2 + 2y^2)$.

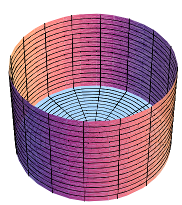
4) (global extrema) Find all extrema of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$ on the plane and characterize them. Do you find a global maximum or global minimum among them?

5) (extrema) The thickness of the region enclosed by the two graphs $f_1(x,y) = 10 - 2x^2 - 2y^2$ and $f_2(x,y) = -x^4 - y^4 - 2$ is denoted by $f(x,y) = f_1(x,y) - f_2(x,y)$. Classify all critical points of $f$ and find the global minimal thickness.



## Section 4.2: Lagrange

1) (Lagrange) Find the cylindrical basket which is open on the top has has the largest volume for fixed area $\pi$. If $x$ is the radius and $y$ is the height, we have to extremize $f(x,y) = \pi x^2 y$ under the constraint $g(x,y) = 2\pi xy + \pi x^2 = \pi$. Use the method of Lagrange multipliers.





2) (global extrema with Lagrange) Find the extrema of the same function

$$f(x,y) = e^{-x^2-y^2}(x^2 + 2y^2)$$

   as in problem 4.1.3 but now on the entire disc $\{x^2 + y^2 \leq 4\}$ of radius 2. Besides the already found extrema inside the disk, you have to find extrema on the boundary.

3) Find and classify all the critical points of the function

$$f(x,y) = 5 + 3x^2 + 3y^2 + y^3 + x^3 .$$

   Is there a global maximum or a global minimum for $f(x,y)$?

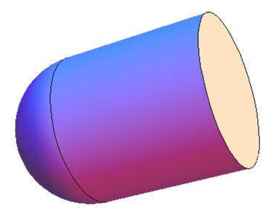
4) A **solid bullet** made of a half sphere and a cylinder has the volume $V = 2\pi r^3/3 + \pi r^2 h$ and surface area $A = 2\pi r^2 + 2\pi r h + \pi r^2$. Doctor Manhatten designs a bullet with fixed volume and minimal area. With $g = 3V/\pi = 1$ and $f = A/\pi$ he therefore minimizes

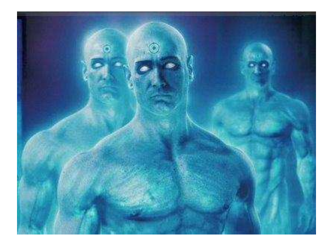
$$f(h,r) = 3r^2 + 2rh$$

   under the constraint

$$g(h,r) = 2r^3 + 3r^2 h = 1 .$$

   Use the Lagrange method to find a local minimum of $f$ under the constraint $g = 1$.

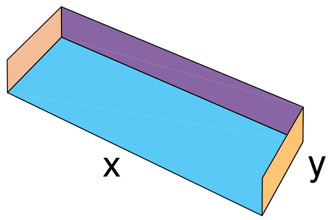




5) Minimize the material cost of an office tray

$$f(x,y) = xy + x + 2y$$

   of length x, width y and height 1 under the constraint that the volume $g(x,y) = xy$ is constant and equal to 4.




## Section 4.3: Double integrals

1) (double integral) Calculate the iterated integral $\int_1^4 \int_0^2 (2x - \sqrt{y})\ dxdy$.

2) (double integral) Find the area of the region

$$R = \{(x,y) \mid 0 \leq x \leq 2\pi, \sin(x) - 1 \leq y \leq \cos(x) + 2\}$$

and use it to compute the average value $\int \int_R f(x,y)\ dxdy/area(R)$ of $f(x,y) = y$ over that region.

3) (volume) Find the volume of the solid lying under the paraboloid $z = x^2 + y^2$ and above the rectangle $R = [-2,2] \times [-3,3] = \{(x,y) \mid -2 \leq x \leq 2, -3 \leq y \leq 3\}$.

4) (switching order of integration) Calculate the iterated integral $\int_0^1 \int_x^{2-x} (x^2 - y)\ dydx$. Sketch the corresponding type I region. Write this integral as integral over a type II region and compute the integral again.

5) (double integral) Evaluate the double integral

$$\int_0^2 \int_{x^2}^4 \frac{x}{e^{y^2}}\ dydx .$$

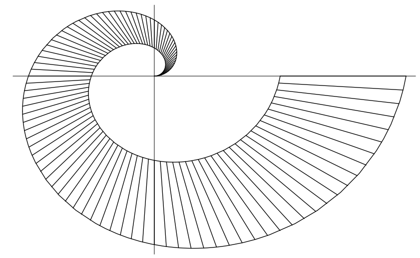
## Section 4.4: Polar integration

1) (polar integrals) Integrate $f(x,y) = x^2$ over the unit disc $\{x^2 + y^2 \leq 1\}$ in two ways, first using Cartesian coordinates, then using polar coordinates.

2) (polar integrals) Find $\int \int_R (x^2+y^2)^{10}\ dA$, where $R$ is the part of the unit disc $\{x^2+y^2 \leq 1\}$ for which $y > x$.



3) (polar integrals) What is the area of the region which is bounded by the following three curves, first by the polar curve $r(\theta) = \theta$ with $\theta \in [0, 2\pi]$, second by the polar curve $r(\theta) = 2\theta$ with $\theta \in [0, 2\pi]$ and third by the positive $x$-axis.



4) (polar integrals) Find the average value of $f(x,y) = x^2 + y^2$ on the annulur region $R : 1 \leq |(x,y)| \leq 2$. The average is $\int_R f\ dxdy / \int_R 1\ dxdy$.

5) (surface area) Find the surface area of the part of the paraboloid $x = y^2 + z^2$ which is inside the cylinder $y^2 + z^2 \leq 9$.