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Homework for Chapter 3. Linearization and Gradient

Section 3.1: Partial derivatives and partial differential equations

1) (partial differential equations) Verify that $u(t,x) = \sin(\cos(t+x))$ is a solution of the transport equation $u_t(t,x) = u_x(t,x)$.

2) (partial differential equations) Verify that $f(x,y) = 3y^2 + x^3$ satisfies the **Euler-Tricomi** partial differential equation $u_{xx} = x u_{yy}$. This partial differential equation is useful in describing **transonic flow**. Can you find an other solution which is not a multiple of the solution given in this problem?

3) (partial differential equations) Verify that $f(x,t) = e^{-rt}\sin(x+ct)$ satisfies the PDE $f_t(x,t) = c f_x(x,t) - r f(x,t)$ called the **advection equation**.

4) (partial derivatives) a) Compute the partial derivatives f_x, f_y of the function $f(x,y) = (xy)^{1/3}$ at $(0,0)$ according to the definition. You have verified that the gradient $\nabla f = \langle f_x, f_y \rangle$ exists at $(0,0)$.
b) The directional derivative $D_{\vec{v}} f$ is defined by $\nabla f \cdot \vec{v}$. Using a), what is the directional derivative at $(0,0)$ if $\vec{v} = \langle 1,1 \rangle/\sqrt{2}$?
c) The chain rule tells us that $\frac{d}{dt} f(t,t) = \nabla f(0,0) \cdot \langle 1,1 \rangle$. You have computed the right hand side in b). But the left hand side of the chain rule is $\frac{d}{dt} f(t,t) = \frac{d}{dt} t^{2/3} = (2/3)t^{-1/3}$ which does not exist at $t=0$. What is going on?

5) (partial differential equations)

The partial differential equation $f_t + f f_x = f_{xx}$ is called Burgers equation and describes waves at the beach. In higher dimensions, it leads to the Navier Stokes equation which are used to describe the weather. Verify that the function

$$f(t,x) \frac{\left(\frac{1}{t}\right)^{3/2} x e^{-\frac{x^2}{4t}}}{\sqrt{\frac{1}{t}} e^{-\frac{x^2}{4t}} + 1}$$

is a solution of the Burgers equation.

Remark. This calculation might need a bit perseverance, when done by hand. You are welcome to use technology if you should get stuck. Here is an example on how to check that a function is a solution of a partial differential equation in Mathematica:

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f[t_,x_]:=(1/Sqrt[t])*Exp[-x^2/(4t)];
Simplify[ D[f[t,x],t] == D[f[t,x],{x,2}]]
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Section 3.2: Linear approximation and tangents

1) (tangent planes) If $2x + 3y + 2z = 9$ is the tangent plane to the graph of $z = f(x,y)$ at the point $(1,1,2)$.
Extimate $f(1.01, 0.98)$

2) (estimation) Estimate $1000^{1/5}$ using linear approximation.

3) (estimation) Find $f(0.01, 0.999)$ for $f(x,y) = \cos(\pi xy)y + \sin(x + \pi y)$

4) (linear approximation Find the linear approximation $L(x,y)$ of the function

$$f(x,y) = \sqrt{10 - x^2 - 5y^2}$$

at $(2,1)$ and use it to estimate $f(1.95, 1.04)$.

5) (tangent lines) Sketch a contour map of the function

$$f(x,y) = x^2 + 9y^2$$

find the **gradient vector** $\nabla f = \langle f_x, f_y \rangle$ of f at the point $(1,1)$. Draw it together with the tangent line $ax + by = d$ to the curve at $(1,1)$.

Section 3.3: Chain rule and implicit differentiation

1) (chain rule) You know that $d/dt f(\vec{r}(t)) = 2$ if $\vec{r}(t) = \langle t, t \rangle$ and $d/dt f(\vec{r}(t)) = 3$ if $\vec{r}(t) = \langle t, -t \rangle$. Find the gradient of f at $(0,0)$.

2) (chain rule) The pressure in the space at the position (x,y,z) is $p(x,y,z) = x^2 + y^2 - z^3$ and the trajectory of an observer is the curve $\vec{r}(t) = \langle t, t, 1/t \rangle$. Using the chain rule, compute the rate of change of the pressure the observer measures at time $t = 2$.

3) (chain rule) Mechanical systems are determined by the energy function $H(x,y)$, a function of two variables. The first, x is the position and the second y is the momentum. The equations of motion for the curve $\vec{r}(t) = \langle x(t), y(t) \rangle$ are

$$\begin{aligned} x'(t) &= H_y(x,y) \\ y'(t) &= -H_x(x,y) \end{aligned}$$

They are called called **Hamilton equations**. a) Using the chain rule, verify that in full generality, the energy of a Hamiltonian system is preserved: for every path $\vec{r}(t) = \langle x(t), y(t) \rangle$ solving the system, we have $H(x(t), y(t)) = const$.
b) Check this in the particular case of the **pendulum**, where $H(x,y) = y^2/2 - \sin(x)$.

4) (implicit differentiation single variable) Derive using implicit differentiation the derivative $d/dx \operatorname{arctanh}(x)$, where $\tanh(x) = \sinh(x)/\cosh(x)$.

   Reminder: the definitions of the **hyperbolic sine** and **hyperbolic cosine** are $\sinh(x) = (e^x - e^{-x})/2$ and $\cosh(x) = (e^x + e^{-x})/2$. Note that $\sinh' = \cosh$ and $\cosh' = \sinh$ and $\cosh^2(x) - \sinh^2(x) = 1$.

5) (implicit differentiation) The equation $f(x,y,z) = e^{xyz} + z = 1 + e$ implicitly defines z as a function $z = g(x,y)$ of x and y. Find formulas (in terms of x,y and z) for $g_x(x,y)$ and $g_y(x,y)$. Estimate $g(1.01, 0.99)$ using linear approximation.

Section 3.4: Gradient and directional derivative

1) (gradient vector) A surface $x^2 + y^2 - z = 1$ radiates light away. It can be parametrized as $\vec{r}(x,y) = \langle x, y, x^2 + y^2 - 1 \rangle$. Find the parametrization of the wave front which is distance 1 from the surface.

2) (directional derivative) Find the directional derivative $D_{\vec{v}} f(2,1) = \nabla f(2,1) \cdot \vec{v}$ into the direction $\vec{v} = \langle -3, 4 \rangle / 5$ for the function $f(x,y) = x^5 y + y^3 + x + y$.

3) (directional derivative) Assume $f(x,y) = 1 - x^2 + y^2$. Compute the directional derivative $D_{\vec{v}}(x,y)$ at $(0,0)$ where $\vec{v} = \langle \cos(t), \sin(t) \rangle$ is a unit vector. Now compute

$$D_v D_v f(x,y)$$

   at $(0,0)$, for any unit vector. For which directions is this **second directional derivative** positive?

4) (gradient) In the following two exercise we derive the so called **Kitchen-Rosenberg formula**

$$\kappa = \frac{f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2}{(f_x^2 + f_y^2)^{3/2}}$$

   for the curvature of a level curve $f(x,y) = c$ at a point (x_0, y_0). It is used in computer vision.Note first that the function

$$g(x,y) = \arctan(f_y/f_x)$$

   is the angle of the gradient vector. Verify that the directional derivative of g in the direction $\vec{v} = \langle -f_y, f_x \rangle / \sqrt{f_x^2 + f_y^2}$ is the curvature.

5) (directional derivative) This is a continuation of the previous problem. Verify that the directional derivative

$$D_{\vec{v}} g(x,y)$$

   computed before is given by the formula above. You have now verified the Kitchen-Rosenberg formula.