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Homework for Chapter 2. Curves and Surfaces

Section 2.1: Functions, level surfaces, quadrics

1) (functions of two variables) Let $f(x,y) = (x^2-1)/(y^2+1)$. Draw the graph. The level surface $S: g(x,y,z) = z - f(x,y) = 0$ of course is the graph of f. Find the equations for the three traces of the surface S. Sketch the contour map.

2) (level surfaces) Consider the surface $z^2 - 4z + x^2 - 2x - y = 0$. Draw the three traces. What surface is it?

3) (level surfaces) Draw the Fermat surface $x^4 + y^4 = z^4$ and its traces.

4) a) Sketch the graph and contour map of the function $f(x,y) = \cos(x^2+y^2)/(1+x^2+y^2)$.
b) Sketch the graph and contour map of the function $g(x,y) = |x| - |y|$.

5) (level surfaces) Verify that the line $\vec{r}(t) = \langle 1,3,2 \rangle + t\langle 1,2,1 \rangle$ is contained in the surface $z^2 - x^2 - y = 0$.

Section 2.2: Parametric surfaces

1) (parametrized surfaces) Plot the surface with the parametrization

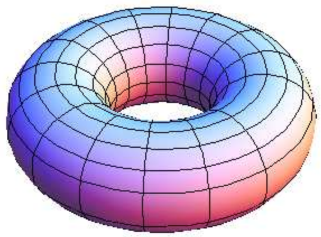
$$\vec{r}(u,v) = \langle v^3\cos(u), v^3\sin(u), v \rangle ,$$

where $u \in [0, 2\pi]$ and $v \in \mathbf{R}$.

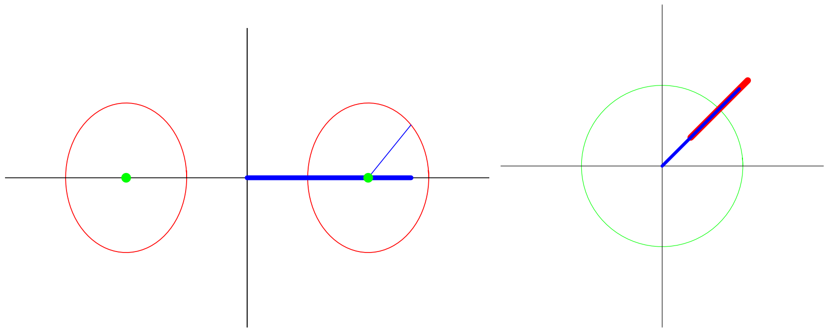
2) (parametrized surfaces) Find a parametrization for the plane which contains the three points $P = (3,7,1)$,$Q = (1,2,1)$ and $R = (0,3,4)$.

3) (parametrized surfaces) a) Find a parametrisations of the lower half of the ellipsoid $2x^2 + 4y^2 + z^2 = 1, z < 0$ by using that the surface is a graph $z = f(x,y)$.
b) Find a second parametrization use angles ϕ, θ similarly as for the sphere.

4) (parametrized surfaces) Find a parametrisation of the **torus** which is obtained as the set of points which have distance 1 from the circle $(2\cos(\theta), 2\sin(\theta), 0)$, where θ is the angle ocuring in cylindrical and spherical coordinates.



Hint: Keep $u = t$ as one of the parameters and let r the distance of a point on the torus to the z-axis. This distance is $r = 2 + \cos(\phi)$ if ϕ is the angle you see on Figure 1. You can read off from the same picture also $z = \sin(\phi)$. To finish the parametrization problem, you have to translate back from cylindrical coordinates $(r,\theta,z) = (2+\cos(\phi), \theta, \sin(\phi))$ to Cartesian coordinates (x,y,z). Write down your result in the form $\vec{r}(\theta,\phi) = \langle x(\theta,\phi), y(\theta,\phi), z(\theta,\phi) \rangle$.



5) (cylindrical and spherical coordinates) a) What is the equation for the surface $x^2 + y^2 - 5x = z^2$ in cylindrical coordinates?
b) Describe in words or draw a sketch of the surface whose equation is $\rho = |\sin(3\phi)|$ in spherical coordinates (ρ, θ, ϕ).

Section 2.3: Parametrized Curves

1) (parametrized curves) Sketch the plane curve $\vec{r}(t) = \langle x(t), y(t) \rangle = \langle t^3, t^2 \rangle$ for $t \in [-1, 1]$ by plotting the points for different values of t. Calculate its velocity $\vec{r}\,'(t)$ as well as its acceleration $\vec{r}\,''(t)$ at the point $t = 2$.

2) (reconstructing curves from acceleration) A device in a car measures the acceleration $\vec{r}\,''(t) = \langle \cos(t), -\cos(3t) \rangle$ at time t. Assume that the car is at the origin $(0, 0)$ at time $t = 0$ and has zero speed at $t = 0$, what is its position $\vec{r}(t)$ at time t?

3) (curves on surfaces) Verify that the curve $\vec{r}(t) = \langle t\cos(t), 2t\sin(t), t^2 \rangle$ is located on the **elliptical paraboloid**
$$z = x^2 + \frac{y^2}{4} \; .$$
Use this fact to sketch the curve.

4) (intersections of surfaces) Find the parameterization $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ of the curve obtained by intersecting the elliptical cylinder $x^2/9 + y^2/4 = 1$ with the surface $z = xy$. Find the velocity vector $\vec{r}\,'(t)$ at the time $t = \pi/2$.

5) (parametrized curves) Consider the curve
$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle = \langle t^2, 1 + t, 1 + t^3 \rangle \; .$$
Check that it passes through the point $(1, 0, 0)$ and find the velocity vector $\vec{r}\,'(t)$, the acceleration vector $\vec{r}\,''(t)$ as well as the jerk vector $\vec{r}\,'''(t)$ at this point.

Section 2.4: Arc length and curvature

1) (arc length) Find the arc length of the curve
$$\vec{r}(t) = \langle t^2, \sin(t) - t\cos(t), \cos(t) + t\sin(t) \rangle \; ,$$
where the time parameter satisfies $0 \leq t \leq \pi$.

2) (curvature) Find the curvature of $\vec{r}(t) = \langle e^t\cos(t), e^t\sin(t), t \rangle$ at the point $(1, 0, 0)$.

3) (unit tangent, normal and bi-normal vectors) Find the vectors $\vec{T}(t), \vec{N}(t)$ and $\vec{B}(t))$ for the curve $\vec{r}(t) = \langle t^2, t^3, 0 \rangle$ for $t = 2$. Do the vectors depend continuously on t for all t?

4) (curvature) Let $\vec{r}(t) = \langle t, t^2 \rangle$. Find the equation for the **caustic**
$$\vec{s}(t) = \vec{r}(t) + \frac{\vec{N}(t)}{\kappa(t)}$$
which is known also as the **evolute** of the curve.

5) (curvature and curves) If $\vec{r}(t) = \langle -\sin(t), \cos(t) \rangle$ is the boundary of a coffee cup and light enters in the direction $\langle -1, 0 \rangle$, then light focuses inside the cup on a curve which is called the **coffee cup caustic**. The light ray travels after the reflection for length $\sin(\theta)/(2\kappa)$ until it reaches the caustic. Find a parameterization of the caustic.