Oliver Knill, Harvard Summer school, Summer 2010

Homework for Chapter 1. Geometry and Space

Section 1.1: Space, distance, geometrical objects

1) (Geometrical objects) Describe and sketch the set of points $P = (x, y, z)$ in three dimensional space $\mathbf{R}^3$ represented by

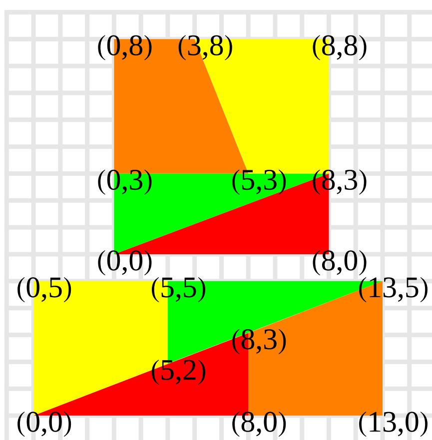
a) $y^2 + z^2 = 1$
b) $x - y - z = 1$
c) $xyz = 0$
d) $x^2 = y$

2) (Distances)

a) Find the distances of $P = (3, 4, 0)$ to all the 3 coordinate axes.
b) Find the distances of $P = (1, 2, 5)$ to all the 3 coordinate planes.

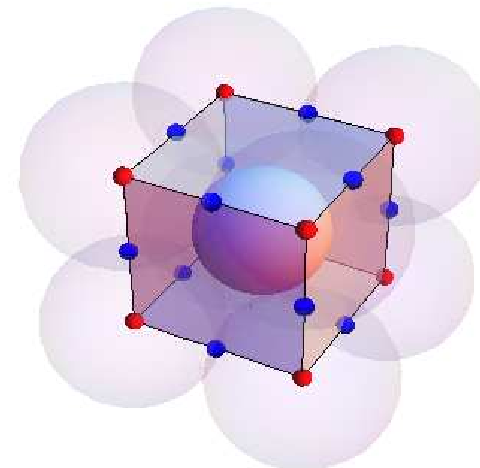
3) (Distances)

Below you see two rectangles. One has the area $8 \cdot 8 = 64$. The other has the area $65 = 13 \cdot 5$. But these triangles are made up by triangles or trapezoids which match. What is going on? Hint. Measure various distances.



4) (Completion of Spheres, traces) Find the center and radius of the sphere $x^2 + 2x + y^2 - 16y + z^2 + 10z + 54 = 0$ Describe the traces of this surface, its intersection with each of the coordinate planes.

5) (Spheres, distances) We place unit spheres on the corners of a unit cube of side length 2 so that adjacent spheres touch. How large is the radius of the sphere in the center of the cube kissing all the 8 spheres?



Section 1.2: Vectors, dot product, projections

1) (Vector operations, length) a) Let $\vec{u} = \langle 2, 3\rangle$ and $\vec{v} = \langle -2, 1\rangle$. Draw $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$.

b) Can you find an inequality between $A = |\vec{u} - \vec{v}|$ and $B = |\vec{u}| - |\vec{v}|$ which holds for all vectors $\vec{u}$ and $\vec{v}$?

2) (Orthogonality, algebra) An **Euler brick** is a cuboid of dimensions a,b,c such that the face diagonals are integers.

a) Verify that $\vec{v} = \langle a, b, c\rangle = \langle 240, 117, 44\rangle$ is a vector which leads to an Euler brick.

b) Verify that $\langle a, b, c\rangle = \langle u(4v^2 - w^2), v(4u^2 - w^2), 4uvw\rangle$ leads to an Euler brick if $u^2 + v^2 = w^2$.

P.S. If also the space diagonal $\sqrt{a^2 + b^2 + c^2}$ is an integer, an Euler brick is called a **perfect cuboid**. It is an open mathematical problem, whether a perfect cuboid exists. Nobody has found one, nor proven that it can not exist.

3) (Angles and projection) **Colors** are encoded by vectors $\vec{v} = \langle red, green, blue\rangle$ The red, green and blue components of $\vec{v}$ are all real numbers in the interval $[0, 1]$. Examples are:

(0,0,0)	black
(1,1,1)	white
(1/2,1/2,1/2)	gray
(1,0,0)	red
(0,1,0)	green
(0,1,1/2)	spring green
(1,1/2,1/2)	pink

(0,0,1)	blue
(1,1,0)	yellow
(1,0,1)	magenta
(0,1,1)	cyan
(1,1/2,0)	orange
(1,1,1/2)	khaki
(1/2,1/4,0)	brown

a) Determine the angle between the colors magenta and cyan.
b) What is the projection of the mixture $(\vec{v}+\vec{w})/2$ of magenta and orange onto blue?

4) (Angles) Find the angle between the diagonal of the unit cube and one of the diagonal of one of its faces. Assume that the two diagonals go through the same edge of the cube. You can leave the answer in the form $\cos(\alpha) = ...$

5) (Length, angle and projection) Assume $\vec{v} = \langle -4, 2, 2\rangle$ and $\vec{w} = \langle 3, 0, 4\rangle$.

a) Find the vector projection of $\vec{v}$ onto $\vec{w}$.
b) Find the scalar component of $\vec{v}$ on $\vec{w}$.

Section 1.3: The cross product and triple scalar product

1) (Cross and triple scalar product) Find the volume of the parallelepiped for which the base parallelogram is given by the points $(0,0,0), (1,0,1, (2,2,1), (1,2,0)$ and which has an edge connecting $(0,0,0)$ with $(1,2,3)$.

2) (Area formula) a) Assume $\vec{u}+\vec{v}+\vec{w} = \vec{0}$. Verify that $\vec{u}\times\vec{v} = \vec{v}\times\vec{w} = \vec{w}\times\vec{u}$.
b) Find $(\vec{u}+\vec{v})\cdot(\vec{v}\times\vec{w})$ if $\vec{u},\vec{v},\vec{w}$ are unit vectors which are orthogonal to each other and $\vec{u}\times\vec{v} = \vec{w}$.

3) (Planes) To find the equation $ax+by+cz=d$ for the plane which contains the point $P=(1,2,3)$ as well as the line which passes through $Q=(3,4,4)$ and $R=(1,1,2)$, we find a vector $\langle a,b,d\rangle$ normal to the plane and fix d so that P is in the plane.

4) (Lagrange formula) Verify the Lagrange formula

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$$

for general vectors $\vec{a},\vec{b},\vec{c}$ in space. The formula can be remembered as "BAC minus CAB".

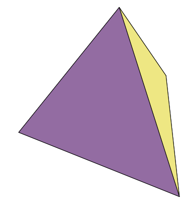
5) (Triple scalar product) Assume you know that the triple scalar product $[\vec{u},\vec{v},\vec{w}] = \vec{u}\cdot(\vec{v}\times\vec{w})$ between $\vec{u},\vec{v},\vec{w}$ is equal to 4. Find the values of $[\vec{v},\vec{u},\vec{w}]$ and $[\vec{u}+\vec{v},\vec{v},\vec{w}]$.

Section 1.4: Lines, planes and distances

1) (Lines and planes) Find the parametric and symmetric equation for the line which passes through the points $P=(1,2,3)$ and $Q=(3,4,5)$.

2) (Distance formula)

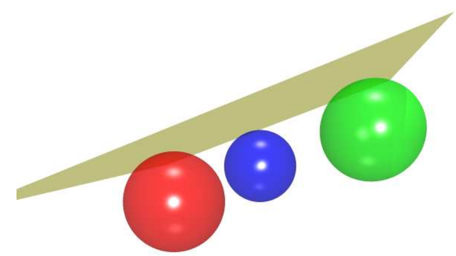
A regular tetrahedron has vertices at the points $P_1 = (0,0,3)$,$P_2 = (0,\sqrt{8},-1)$, $P_3 = (-\sqrt{6},-\sqrt{2},-1)$ and $P_4 = (\sqrt{6},-\sqrt{2},-1)$. Find the distance between two edges which do not intersect.



3) (Geometric constructions) Find a parametric equation for the line through the point $P=(3,1,2)$ that is perpendicular to the line $L: x=1+t, y=1-t, z=2t$ and intersects this line in a point Q.

4) (Geometric constructions) Given three spheres of radius 1 centered at $A=(1,2,0), B=(4,5,0), C=(1,3,2)$. Find a plane $ax+by+cz=d$ which touches all of three spheres from the same side.

Hint. There are two such planes. You want to consider the plane through the three points A,B,C first. You only need to find one of the two possible planes touching all the spheres on the same side.



5) (Distance formulas)
a) Find the distance between the point $P=(3,3,4)$ and the line $x=y=z$.
b) Parametrize the line $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ in a) and find the minimum of the function $f(t) = d(P,\vec{r}(t))^2$. Verify that the the minimal value agrees with a).