

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) (20 points)

- 1) ☐ T ☐ F The line $\vec{r}(t) = \langle t, t, t \rangle$ is perpendicular to the plane $x + y + z = 10$.

Solution:

Yes, the normal vector to the plane is $\langle 1, 1, 1 \rangle$.

- 2) ☐ T ☒ F The quadratic surface $-x^2 + y^2 + z^2 = -1$ is a one sheeted hyperboloid.

Solution:

It is a two sheeted hyperboloid.

- 3) ☐ T ☒ F The relation $|\vec{u} \times \vec{v}| = |\vec{u} \cdot \vec{v}|$ is only possible if at least one of the vectors $\vec{u}$ and $\vec{v}$ is the zero vector.

Solution:

It is also possible if they are nonzero but form a 45 degree angle.

- 4) ☐ T ☒ F $\int_0^{\pi/2} \int_0^1 r^3 d\theta dr = \int_0^1 \int_0^1 x^2 + y^2 dx dy$.

Solution:

The bounds are wrong. The second integral integrates over a square.

- 5) ☐ T ☒ F If a vector field $\vec{F}(x, y)$ satisfies $\text{curl}(\vec{F})(x, y) = Q_x(x, y) - P_y(x, y) = 0$ and $\text{div}(\vec{F})(x, y) = P_x(x, y) + Q_y(x, y) = 0$ for all points (x, y) in the plane, then $\vec{F}$ is a constant field.

Solution:

There are more fields which satisfy this. An example is $\vec{F}(x, y) = \langle x + y, x - y \rangle$.

- 6) ☐ T ☒ F The acceleration vector $\vec{r}''(t) = \langle x(t), y(t) \rangle$, the velocity vector $\vec{r}'(t)$ and $\vec{r}'(t) \times \vec{r}''(t)$ form three vectors which are mutually perpendicular.

Solution:

Already for a straight line, this is not the case.

- 7)

T	F
---	---

 The curvature of the curve $\vec{r}(t) = \langle \sin(2t), 0, \cos(2t) \rangle$ is equal to the curvature of the curve $\vec{s}(t) = \langle 0, \cos(3t), \sin(3t) \rangle$.

Solution:

Both are circles of radius 1.

- 8)

T	F
---	---

 The space curve $\vec{r}(t) = \langle t \sin(t), t \cos(t), t^2 \rangle$ for $t \in [0, 10\pi]$ is located on a cone.

Solution:

It is not $x(t)^2 + y(t)^2 = z(t)^2$ but $x(t)^2 + y(t)^2 = z(t)$ so that it is located on a paraboloid.

- 9)

T	F
---	---

 If a smooth function $f(x, y)$ has a global maximum, then this maximum is a critical point.

Solution:

It is then also a local maximum.

- 10)

T	F
---	---

 If $L(x, y)$ is the linearization of $f(x, y)$ and $\vec{s}(t)$ is the line tangent to the curve $\vec{r}(t)$ at t_0 . Then $d/dt L(\vec{s}(t)) = d/dt f(\vec{r}(t))$ at the time $t = t_0$.

Solution:

This is how the chain rule can be proved.

- 11)

T	F
---	---

 If $\vec{F}$ is a gradient field and $\vec{r}(t)$ is a flow line defined by $\vec{r}'(t) = \vec{F}(\vec{r}(t))$, then the line integral $\int_0^1 \vec{F} \cdot d\vec{r}$ is either positive or zero.

Solution:

The power is positive.

- 12)

T	F
---	---

 If we extremize the function $f(x, y)$ under the constraint $g(x, y) = 1$, and the functions are the same $f = g$, we have infinitely many extrema.

Solution:

Every point on the curve $g(x, y) = 1$ is a solution to the Lagrange equations.

- 13)

T	F
---	---

 If a point (x_0, y_0) is a critical point of $f(x, y)$ under the constraint $g(x, y) = 1$, then it is also a critical point of the function $f(x, y)$ without constraints.

Solution:

The gradient of f does not have to be the zero vector.

- 14)

T	F
---	---

 If a vector field $\vec{F}(x, y)$ is a gradient field, then any line integral along any ellipse is zero.

Solution:

This follows from the fundamental theorem of line integrals.

- 15)

T	F
---	---

 The flux of an irrotational vector field is zero through any surface S in space.

Solution:

It is only zero through a closed surface.

- 16)

T	F
---	---

 The divergence of a gradient field $\vec{F}(x, y) = \nabla f(x, y)$ is zero.

Solution:

While the curl of a gradient is zero and the divergence of a curl is zero, the divergence of a gradient is the Laplacian of f and not necessarily zero.

- 17)

T	F
---	---

 The line integral of a vector field $\vec{F}(x, y, z) = \langle x, y, z \rangle$ along a circle in the xy - plane is zero.

Solution:

Yes, by the fundamental theorem of line integrals.

- 18)

T	F
---	---

 For any solid E , the moment of inertia $\iiint_E x^2 + y^2 \, dxdydz$ is always larger than the volume $\iiint_E 1 \, dxdydz$.

Solution:

For small solids, the moment of inertia is small, for large solids, the moment of inertia is large.

- 19)

T

F

 The curvature of a circle is always larger than the acceleration.

Solution:

The acceleration depends on the parametrization.

- 20)

T

F

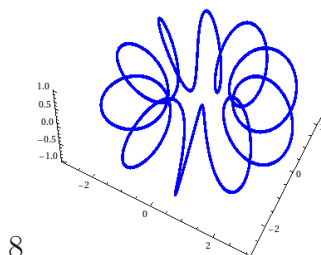
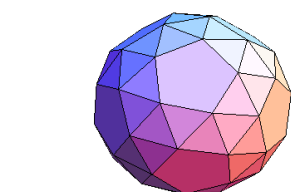
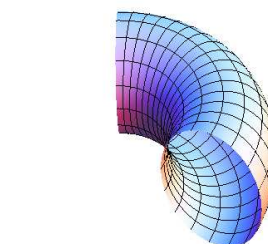
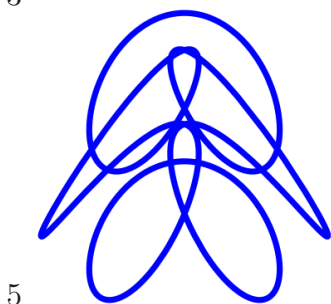
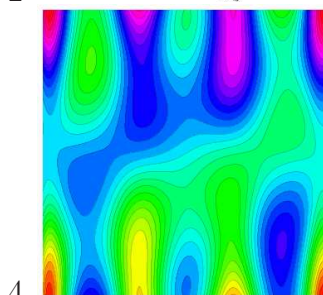
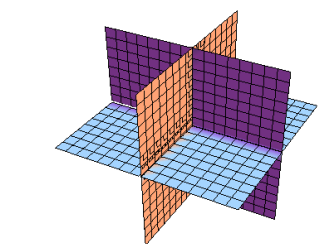
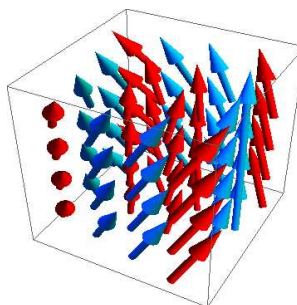
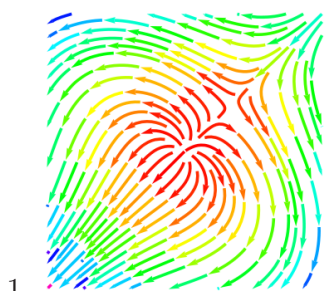
 The directional derivative of $\text{div}(\vec{F}(x, y))$ of the divergence of the vector field $\vec{F} = \langle P, Q \rangle$ in the direction $\vec{v} = \langle 1, 0 \rangle$ is $P_{xx} + Q_{xy}$.

Solution:

Yes by definition $\text{div}(\vec{F}(x, y)) = P_x(x, y) + Q_y(x, y)$. The directional derivative in the $\langle 1, 0 \rangle$ direction is the partial derivative.

Problem 2) (10 points) Match objects with definitions. No justifications necessary.

Match the objects with their definitions



Enter 1-8 or 0 if no match	Object definition
	$\vec{r}(t) = \langle (2 + \cos(10t)) \cos(t), (2 + \cos(10t)) \sin(t), \sin(10t) \rangle$
	$\vec{F}(x, y, z) = \langle -y, x, 2 \rangle$
	$\vec{r}(t, s) = \langle (2 + \cos(s)) \cos(t), (2 + \cos(s)) \sin(t), \sin(s) \rangle$
	$\{(x, y, z) \mid \sin(x^2) - \cos(y^2) = 1\}$
	$\vec{F}(x, y) = \langle x - y^2, y - x^2 \rangle$
	$xyz = 0$
	$x^2 + y^2 - z^2 = 1$
	$\{(x, y) \mid \sin(x^2 \sin(x))y + \sin(y - x) = c\}$
	$\vec{r}(t) = \langle \sin(t) + \cos(5t), \cos(t) + \cos(6t) \rangle$

Solution:

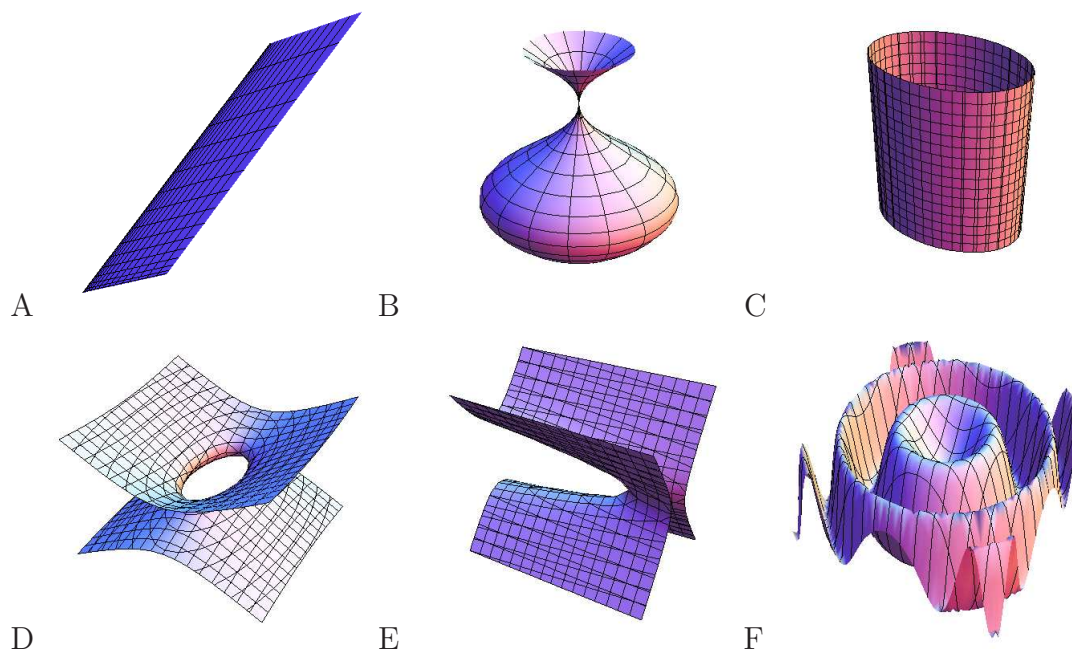
8,2,6,0,1,3,0,4,5

Problem 3) (10 points)

a) (4 points) Check every box to the left, for which the missing part to the right is $\nabla f(1, 2)$. The function $f(x, y)$ is an arbitrary nice function like for example $f(x, y) = x - yx + y^2$. The curve $\vec{r}(t)$, wherever it appears, parametrizes the level curve $f(x, y) = f(1, 2)$ and has the property that $\vec{r}(0) = \langle 1, 2 \rangle$.

Check	Topic	Statement
☐	Linearization	$L(x, y) = f(1, 2) + \boxed{} \cdot \langle x - 1, y - 2 \rangle$
☐	Chain rule	$\frac{d}{dt} f(\vec{r}(t)) _{t=0} = \boxed{} \cdot \vec{r}'(0)$
☐	Steepest descent	f decreases at $(1, 2)$ most in the direction of $\boxed{}$
☐	Estimation	$f(1 + 0.1, 1.99) \sim f(1, 2) + \boxed{} \cdot \langle 0.1, -0.01 \rangle$
☐	Directional derivative	$D_{\vec{v}} f(1, 2) = \boxed{} \cdot \vec{v}$
☐	Level curve	of f through $(1, 2)$ has the form $\boxed{} \cdot \langle x - 1, y - 2 \rangle = 0$
☐	Vector projection	of $\nabla f(1, 2)$ onto $\vec{v}$ is $\vec{v}(\vec{v} \cdot \boxed{}) / \vec{v} ^2$
☐	Tangent line	of $\vec{r}(t)$ at $(1, 2)$ is parametrized by $\vec{R}(s) = \langle 1, 2 \rangle + s \boxed{}$

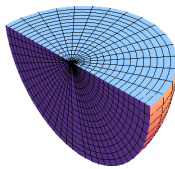
b) (3 points) The surfaces are given either as a parametrization or implicitly. Match them. Each surface matches one definition.



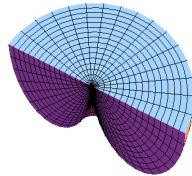
Enter A-F here	Function or parametrization
	$\vec{r}(u, v) = \langle u^2, v^2, u^2 + v^2 \rangle$
	$\vec{r}(u, v) = \langle (1 + \sin(u)) \cos(v), (1 + \sin(u)) \sin(v), u \rangle$
	$4x^2 + y^2 - 9z^2 = 1$
	$x - 9y^2 + 4z^2 = 1$
	$\vec{r}(u, v) = \langle u, v, \sin(u^2 + v^2) \rangle$
	$4x^2 + 9y^2 = 1$

c) (3 points) Match the solids with the triple integrals:

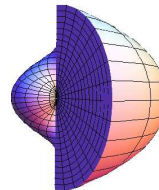
Enter A-D	3D integral computing volume
	$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{1/\cos(\phi)} \rho^2 \sin(\phi) \, d\rho d\phi d\theta$
	$\int_0^\pi \int_{\pi/2}^\pi \int_0^{\sin(\phi)} \rho^2 \sin(\phi) \, d\rho d\phi d\theta$
	$\int_0^\pi \int_{\pi/2}^\pi \int_0^1 \rho^2 \sin(\phi) \, d\rho d\phi d\theta$
	$\int_0^{2\pi} \int_0^\pi \int_0^\theta \rho^2 \sin(\phi) \, d\rho d\phi d\theta$



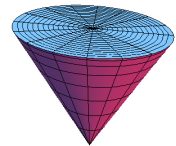
A



B



C



D

Solution:

a) Everything except 3,6,8.

b) A,B,D,E,F,C

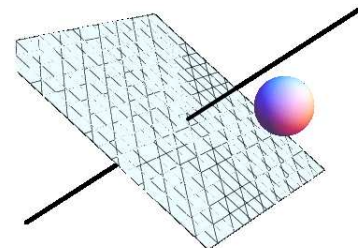
c) D,B,A,C

Problem 4) (10 points)

We want to determine whether the distance of the sphere S of radius 1 centered at $P = (1, 2, 3)$ to the plane $E : x + y + z = 1$ is larger than the distance of the same sphere to the line $L : x + y = y + z = x + z$.

a) (5 points) Find the distance from the sphere S to the plane E .

b) (5 points) Find the distance from the sphere S to the line L .



Solution:

a) The point $Q = (1, 0, 0)$ is on the plane with normal vector $\langle 1, 1, 1 \rangle$. The distance between P and the plane is

$$d(P, E) = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|} = \frac{|\langle 0, 2, 3 \rangle \cdot \langle 1, 1, 1 \rangle|}{\sqrt{3}} = \frac{5}{\sqrt{3}}.$$

The distance between the sphere and the plane is $\boxed{\frac{5}{\sqrt{3}} - 1}$.

b) The distance between P and the line L which contains the point $R = \langle 0, 0, 0 \rangle$ and the vector $\langle 1, 1, 1 \rangle$ is

$$d(P, L) = \frac{|\vec{PR} \times \langle 1, 1, 1 \rangle|}{\sqrt{3}} = \frac{|\langle 1, 2, 3 \rangle \times \langle 1, 1, 1 \rangle|}{\sqrt{3}} = \frac{|\langle 1, -2, 1 \rangle|}{\sqrt{3}} = \sqrt{6}/\sqrt{3} = \sqrt{2}$$

The distance between the sphere and the line is $\boxed{\sqrt{2} - 1}$. The distance to the cylinder is much smaller.

Problem 5) (10 points)

Where does the vector field

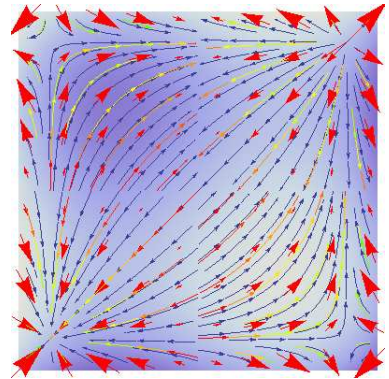
$$\vec{F}(x, y) = \langle P, Q \rangle = \langle y(x^3 - 3x), x(y^3 - 3y) \rangle$$

have maximal or minimal curl

$$f(x, y) = \text{curl}(\vec{F})(x, y) = Q_x(x, y) - P_y(x, y).$$

a) (8 points) Find all extrema and determine whether they are maxima, minima or saddle points.

b) (2 points) Is there a global maximum of $f(x, y)$?



Solution:

The function is $f(x, y) = y^3 - 3y - (x^3 - 3x)$. In order to find the critical points, we look for solutions of $\langle -3x^2 + 3, 3y^2 - 3 \rangle = \langle 0, 0 \rangle$. It is zero at $x = \pm 1, y = \pm 1$. Lets make a table to apply the second derivative test:

x	y	D	f_{xx}	Type	f(x,y)
-1	-1	-36	6	saddle	0
-1	1	36	6	minimum	-4
1	-1	36	-6	maximum	4
1	1	-36	-6	saddle	0

b) No, there is no global maximum nor a global minimum. Already restricting the function to the x axes shows that its range is the entire z axes.

Problem 6) (10 points)

A sprinkler at position $(0, 0, 1)$ throws out water with constant speed and elevation angle 45 degrees. The water is under constant gravitational acceleration $\langle 0, 0, -10 \rangle$.

a) (5 points) Find the trajectory $\vec{r}(t)$, if the initial velocity is $\vec{r}'(t)|_{t=0} = \langle \cos(\theta), \sin(\theta), 1 \rangle$ and write down the formula for the arc length from $t = 0$ to $t = 1$. You do not have to start evaluating the integral.

b) (5 points) All the trajectories together form a surface $\vec{r}(\theta, t)$. Parametrize this surface and write down the formula for the surface area if $0 \leq t \leq 1$ and $0 \leq \theta \leq 2\pi$. You do not have to start evaluating the integral.



Solution:

a) We integrate

$$\vec{r}''(t) = \langle 0, 0, -10 \rangle$$

and use the initial velocity to get

$$\vec{r}'(t) = \langle 0, 0, -10t \rangle + \langle \cos(\theta), \sin(\theta), 1 \rangle .$$

Integrate again using the initial position $\langle 0, 0, 1 \rangle$ to get $\vec{r}(t) = \langle \cos(\theta)t, \sin(\theta)t, 1 + t - 5t^2 \rangle$.
To get the arc length, we use the formula

$$\int_0^1 |\vec{r}'(t)| dt = \int_0^1 \sqrt{1 + (1 - 10t)^2} dt .$$

b) The parametrization of the surface is

$$\vec{r}(\theta, t) = \langle \cos(\theta)t, \sin(\theta)t, 1 + t - 5t^2 \rangle .$$

To get the surface area

$$\int_0^{2\pi} \int_0^1 |\vec{r}_t \times \vec{r}_\theta| dt ,$$

compute

$$\vec{r}_t \times \vec{r}_\theta = \langle \cos(\theta), \sin(\theta), 1 - 10t \rangle \times \langle -\sin(\theta)t, \cos(\theta)t, 0 \rangle = \langle t(10t - 1) \cos(\theta), t(10t - 1) \sin(\theta), t \rangle$$

and its length

$$t\sqrt{(10t - 1)^2 + 1}$$

go get

$$\int_0^{2\pi} \int_0^1 t\sqrt{(10t - 1)^2 + 1} dt d\theta .$$

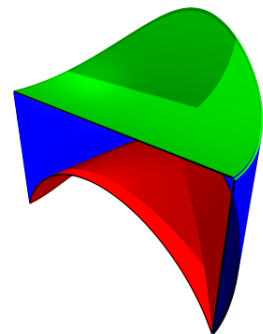
Problem 7) (10 points)

Compute the integral

$$\iiint_E x^2 dz dx dy ,$$

over the solid E defined by the four conditions

$$x^2 + y^2 \leq 1, y \geq 0, z < 4 - y^2, \text{ and } z > -5 + x^2 .$$



Solution:

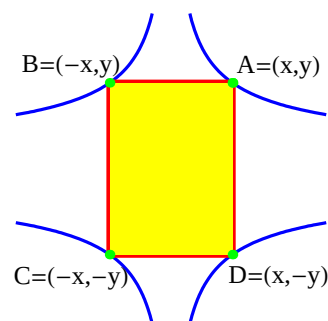
It is important to set up the integral in **cylindrical coordinates**. This can be done directly or could also be done by first defining the integral in cartesian coordinates and then move to cylindrical coordinates. It is

$$\int_0^1 \int_0^\pi r^3 \cos^2(\theta)(9 - r^2) d\theta dr .$$

The result is $\frac{\pi}{2}(9/4 - 1/6) = \boxed{\pi \frac{25}{24}}$.

Problem 8) (10 points)

Use the method of Lagrange multipliers to find the centrally symmetric rectangle with corners $A = (x, y)$, $B = (-x, y)$, $C = (-x, -y)$, $D = (x, -y)$ on the curve $g(x, y) = x^2 y^4 = 1$ which has minimal circumference $f(x, y) = 4x + 4y$.

**Solution:**

We have to extremize the function $f(x, y) = 4x + 4y$ under the constraint $g(x, y) = x^2 y^4 = 1$. The Lagrange equations are

$$\begin{aligned} 4 &= \lambda 2xy^4 \\ 4 &= \lambda 4x^2 y^3 \\ x^2 y^4 &= 1 . \end{aligned}$$

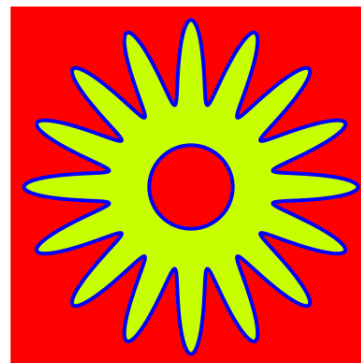
Dividing the first equation by the second is possible because x, y can not be zero. We get $2x = y$. Plugging this into the third equation gives $16x^6 = 1$ or $x = (1/16)^{1/6} = 1/4^{1/3}, y = 2/4^{1/3}$.

Problem 9) (10 points)

a) (5 points) Find the area of the region which is given in polar coordinates (r, θ) as

$$1 \leq r \leq 2 + \cos(16\theta) .$$

The picture of this region can be admired to the right.



b) (5 points) Find

$$\int_0^{\pi/2} \int_x^{\pi/2} \sin(y)/y \, dy dx .$$

Solution:

a) This is already given in polar coordinates. So let's set up the integral in polar coordinates

$$\int_0^{2\pi} \left((2 + \cos(16\theta))^2 - 1 \right) / 2 \, d\theta$$

It evaluates to $\boxed{7\pi/2}$.

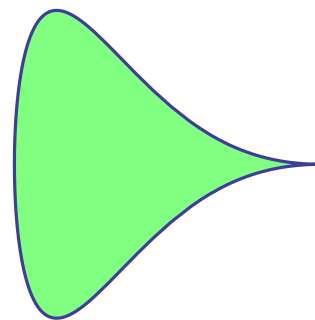
b) We have to do a change the order of integration: $\int_0^{\pi/2} \int_0^y \sin(y)/y \, dx dy = \boxed{1}$.

Problem 10) (10 points)

Find the area of the region enclosed by

$$\vec{r}(t) = \left\langle t^2, \frac{(\sin(\pi t))^2}{t} \right\rangle$$

for $-1 \leq t \leq 1$. Use an integral theorem with a suitable vector field.



Solution:

We use Green's theorem. While it was possible to use $\vec{F} = \langle 0, x \rangle$, the computation is much simpler with $\vec{F} = \langle -y, 0 \rangle$. The line integral of the given curve is

$$\int_{-1}^1 \left\langle -\frac{\sin(\pi t)^2}{t}, 0 \right\rangle \cdot \langle 2t, \dots \rangle dt = \int_{-1}^1 -2 \sin(\pi t)^2 dt = -2.$$

Note that the curve passes around the region in the **clockwise** direction so that we have to take the opposite sign. The area is $\boxed{2}$.

Problem 11) (10 points)

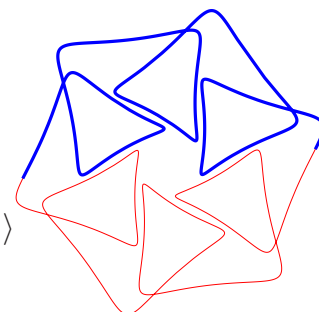
Compute the line integral of the vector field

$$\vec{F}(x, y) = \langle 3x + 2xy^2, 2y + 2x^2y \rangle$$

along the curve

$$\vec{r}(t) = \langle 6 \cos(t) + 4 \cos(7t) + \sin(17t), 6 \sin(t) + 4 \sin(7t) + \cos(17t) \rangle$$

from $t = 0$ to $t = \pi$.

**Solution:**

Use the fundamental theorem of line integrals. The curve is at $t = 0$ at $\langle 10, 1 \rangle$ and at $t = 1$ at $\langle -10, -1 \rangle$. The curve is not closed. The vector field $\vec{F}(x, y, z)$ has the potential $f(x, y, z) = x^2 + y^2 + x^2y^2$. The result (as given by the FTL) is $f(-10, -1) - f(10, 1) = 201 - 201 = 0$. The answer is $\boxed{0}$.

Problem 12) (10 points)

Find the flux of vector field

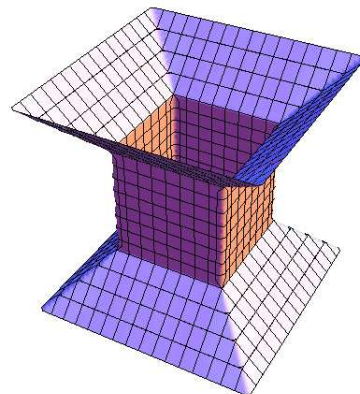
$$\vec{F}(x, y, z) = \langle y^2 \sin(z) - x, z^2 \cos(x^2), 5z \rangle$$

through the surface S given by the $p \rightarrow \infty$ limit of

$$|x|^p + |y|^p - |z|^p = 1, \quad -2 < z < 2.$$

The surface is oriented that the normal vectors points outwards.

Hints. The surface (see picture) becomes closed if two not yet included square "lids" at $z = 2$ and $z = -2$ with corners at $(\pm 2, \pm 2, \pm 2)$ are added. You can use without proof that the volume of the solid is $(16 + 4)/2 + (16 + 4)/2 + 2 * 2 = 20 + 4 = 24$.



Solution:

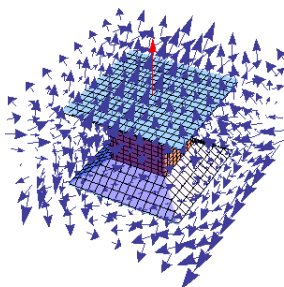
We use the divergence theorem. The divergence of the vector field is constant 4. The flux through the surface S plus the flux through the two lids S_1, S_2 is according to this theorem equal to the volume times the divergence:

$$96 = \iiint_E \operatorname{div}(\vec{F}) \, dx dy dz = \iint_S \vec{F} \cdot d\vec{S} + \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S}.$$

On the upper lid S_1 , the vector field $\vec{F}(\vec{r}(u, v))$ dotted with the normal vector $\vec{r}_u \times \vec{r}_v = \langle 0, 0, 1 \rangle$ of the upper lid $\vec{r}(u, v) = \langle u, v, 2 \rangle$

$$\vec{F}(\vec{r}(u, v)) \cdot \vec{r}_u \times \vec{r}_v = 5z = 10.$$

The flux through the upper lid is therefore $16 * 10 = 160$. Similarly the flux through the lower lid can be computed. Note that the normal vector points downwards on the lower side of the solid so that the flux is also 160 through the lower lid. The flux through



S is $96 - 160 - 160 = \boxed{-224}$.

[Since other numbers were given as hints in the exam, any numerical discrepancies in this problem were discarded during grading. Using the divergence theorem correctly and indicating how to get the flux through the lids as well as how to add things up would give full credit.]

Problem 13) (10 points)

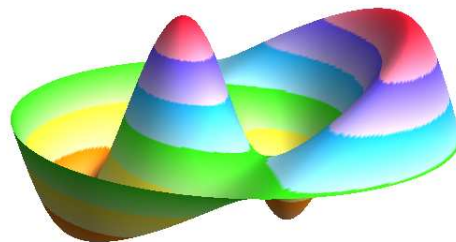
Compute the flux of the curl of the vector field

$$\vec{F}(x, y, z) = \langle x^2, y, \sin(z^2)^4 \rangle$$

through the surface which has the parametrization

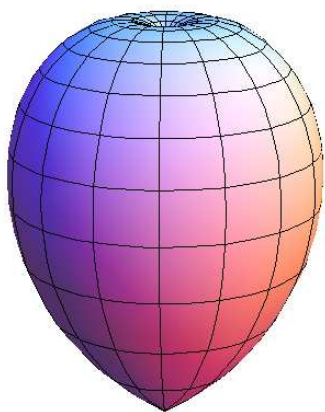
$$\vec{r}(t, s) = \langle s \cos(t), s \sin(t), 3 \sin(s) \cos(t) \rangle ,$$

where $0 \leq t \leq 2\pi$ and $0 \leq s \leq 2\pi$.



Solution:

Since the curl of the vector field is constant $\langle 0, 0, 0 \rangle$, the flux must be zero. But one can also use Stokes theorem and relate the flux as the line integral along the boundary of the surface. The boundary is a circle of radius 3 in the xy plane parametrized by $\langle 2\pi \cos(t), 2\pi \sin(t), 0 \rangle$. This line integral is zero by direct computation. One could also invoke the fundamental theorem of line integral here. Because the curl is zero, the vector field is a gradient field and has therefore the closed loop property so that the line integral along any closed curve is zero. The answer is 0.



X-Rays have intensity and direction and are given by a vector field

$$\vec{F}(x, y, z) = \langle z^7, \sin(z) + y + z^{77}, z + \cos(xy) + \sin(y) \rangle .$$

A **tonsil** is given in spherical coordinates as $\rho \leq \phi$. Find the flux of the X-Ray field $\vec{F}$ through the surface $\rho = \phi$ of the tonsil. The surface is oriented with normal vectors pointing outside. **Remark:** The flux is the amount of **ionizing radiation** absorbed by the tissue. This X-ray exposure is measured in the unit **Gray** which corresponds to the radiation amount to deposit 1 **joule** of energy in 1 **kilogram** of matter and corresponds to about 100 **Rem**. A typical dental X-ray is reported to lead to about one tenth to one half of a Rem.

Solution:

The divergence of the vector field is constant 2. The flux through the tonsil surface is therefore by the divergence theorem 2 times the volume of the tonsil. This volume integral is best done in **spherical coordinates**:

$$2 \int_0^{2\pi} \int_0^\pi \int_0^\phi \sin(\phi) \rho^2 \sin(\phi) \, d\rho d\phi d\theta = 4\pi \int_0^\pi \phi^3 \sin(\phi) \, d\phi .$$

This integral needs integration by parts:

$$\begin{aligned} 2 \text{ Volume(tonsil)} &= \frac{4\pi}{3} [-\phi^3 \cos(\phi)|_0^\pi + \int_0^\pi 3\phi^2 \cos(\phi) \, d\phi] \\ &= \frac{4\pi}{3} [\pi^3 - 3 \int_0^\pi \sin(\phi) 2\phi \, d\phi] \\ &= \frac{4\pi}{3} [\pi^3 + 6\phi \cos(\phi)|_0^\pi - 6 \int_0^\pi \cos(\phi) \, d\phi] \\ &= \frac{4\pi}{3} (\pi^3 - 6\pi) . \end{aligned}$$

The computation could be simplified a bit by switching the order of integration (for the triangle in the ϕ, ρ plane) requiring less integration by parts. The result is still

$$\boxed{\frac{4\pi}{3} (\pi^3 - 6\pi)} .$$