7/9/2009 FIRST HOURLY PRACTICE III Maths 21a, O.Knill, Summer 2009

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
Total:		120

Problem 1) (20 points) No justifications are needed.

1)	T	**F**	The vector $\vec{v} = (1, 2, -4)$ is perpendicular to the plane $4x + 2y + 2z = 100$.
2)	**T**	F	With $\vec{i} = (1, 0, 0), \vec{j} = (0, 1, 0), \vec{k} = (0, 0, 1)$, the formula $(\vec{i} \times \vec{j}) \times (\vec{j} \times \vec{i}) = \vec{0}$ holds.
3)	**T**	F	The equations $x/5 = y/7 = z/8$ describe a line which contains the origin $(0, 0, 0)$.
4)	T	**F**	The vectors $\vec{u} = (3, -2, 1)$ and $\vec{OQ}$ with $O = (0, 0, 0)$ and $Q = (-6, 4, 2)$ are parallel.
5)	**T**	F	If $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ are perpendicular, then the vectors $\vec{u}$ and $\vec{v}$ have the same length.
6)	**T**	F	The two vectors $(2, 3, 0)$ and $(6, -4, 5)$ are orthogonal.
7)	T	**F**	For any two vectors $\vec{v}, \vec{w}$, one has $\lvert\vec{v} + \vec{w}\rvert^2 = \lvert\vec{v}\rvert^2 + \lvert\vec{w}\rvert^2$.
8)	**T**	F	The surface $x^2 - y^2 + z^2 = 1$ is called a one-sheeted hyperboloid.
9)	**T**	F	The set of points which have distance 2 from the x-axis is a cylinder.
10)	**T**	F	If in spherical coordinates a point is given by $(\rho, \theta, \phi) = (2, \pi/2, \pi/2)$, then its Euclidean coordinates is $(x, y, z) = (0, 2, 0)$.
11)	T	**F**	The triangle defined by the three points $(-1, 0, 2), (-4, 2, 1), (1, -1, 2)$ has a right angle.

Solution:
It would be inefficient to compute the angles. Better is to look at the squares of the lengths of the triangle which are $14, 5$ and 35. If there was a right angle, Pythagoras would apply.

12)	**T**	F	A surface which is given as $r = 2 + \sin(z)$ in cylindrical coordinates stays the same when we rotate it around the z axis.
13)	T	**F**	The length $\lvert\vec{v} - \vec{w}\rvert$ of the difference $\vec{v} - \vec{w}$ of two parallel vectors $\vec{v}, \vec{w}$ is always equal to the difference $\lvert\vec{v}\rvert - \lvert\vec{w}\rvert$ of the lengths of the vectors.
14)	T	**F**	The volume of a parallelepiped spanned by $(1, 0, 0), (0, 1, 0)$ and $(1, 1, 1)$ is equal to $1/3$.
15)	T	**F**	The equation $x^2 = y + z^2$ describes an elliptic paraboloid.
16)	**T**	F	In spherical coordinates, the equation $\cos(\theta) = \sin(\theta)$ is the plane $x - y = 0$.

17) T F (F highlighted) If $|\vec{v} \times \vec{w}| = 0$ then $\vec{v} = \vec{0}$ or $\vec{w} = \vec{0}$ or $\vec{v} = \vec{w}$.

18) T F (T highlighted) The vector projection of the vector $(1, 1, 1)$ onto the vector $(0, 2, 0)$ is $(0, 1, 0)$.

19) T F (F highlighted) The point given in spherical coordinates as $(\rho, \theta, \phi) = (\sqrt{8}, 3\pi/2, \pi/2)$ is in Cartesian coordinates the point $(x, y, z) = (2, -2, 0)$.

20) T F (F highlighted) If $g(x, y, z) = 0$ is an implicit equation, then $\vec{r}(u, v) = (u, v, g(u, v, g(u, v, 1)))$ is a parametrization of the surface.

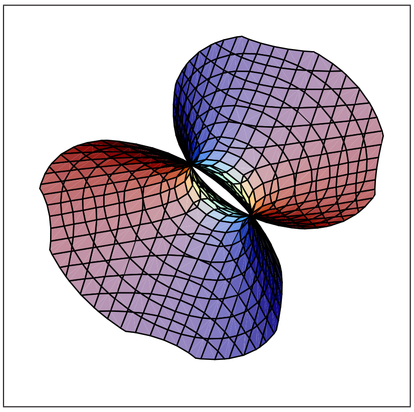
Solution:
Check with a random function g like $g(x, y, z) = x$ to see that it can not work.

	Total

Problem 2) (10 points)

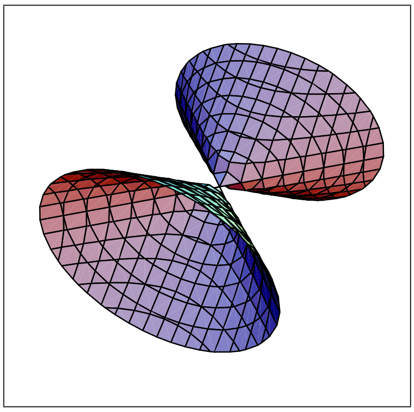
Match the equations $g(x, y, z) = d$ with the surfaces.

I



IV

II



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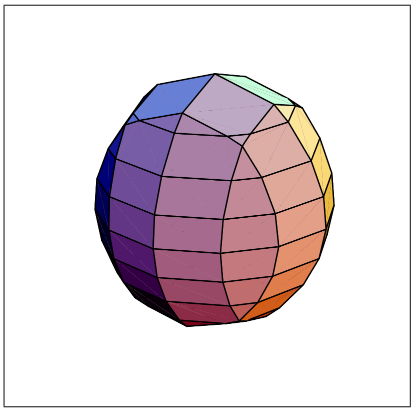
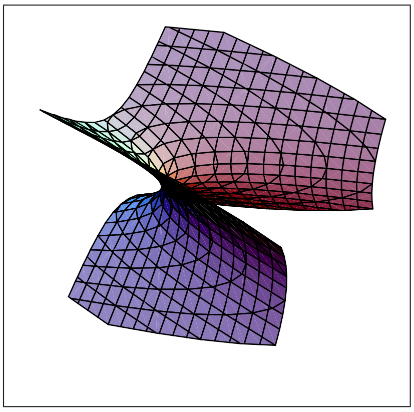
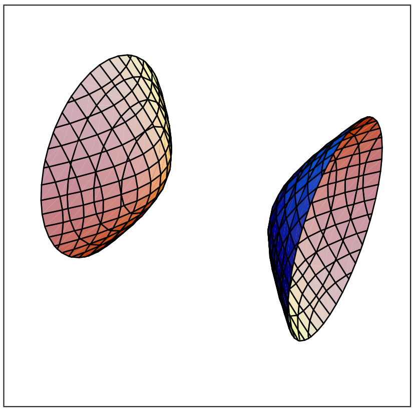
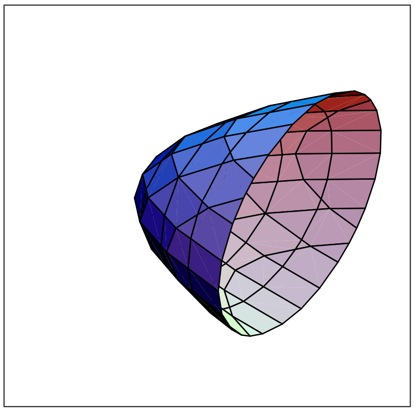
III



VI

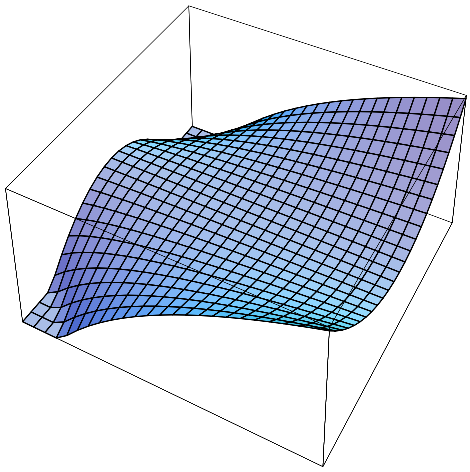
Enter I,II,III,IV,V,VI here	Equation
	$x^2 + 2z^2 - y^2 = 0$
	$y^2 + z^2 + 4x^2 = 1$
	$y^2 - z^2 - x^2 = -1$
	$x^2 - y^2 - z^2 = 1$
	$y^2 - z^2 - x = 1$
	$-y^2 - z^2 + x = 1$

Solution:

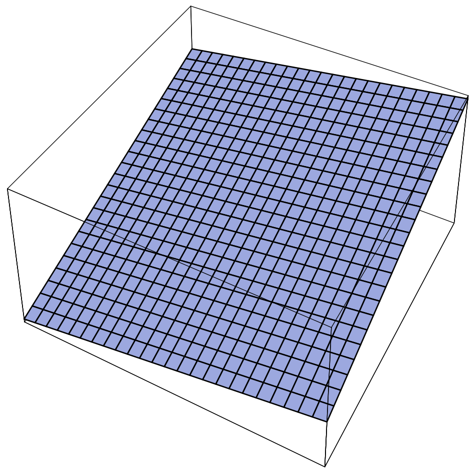
Enter I,II,III,IV,V,VI here	Equation
II	$x^2 + 2z^2 - y^2 = 0$
III	$y^2 + z^2 + 4x^2 = 1$
I	$y^2 - z^2 - x^2 = -1$
V	$x^2 - y^2 - z^2 = 1$
IV	$y^2 - z^2 - x = 1$
VI	$-y^2 - z^2 + x = 1$

Problem 3) (10 points)

Match the equation with their graphs. No justifications are needed.



I



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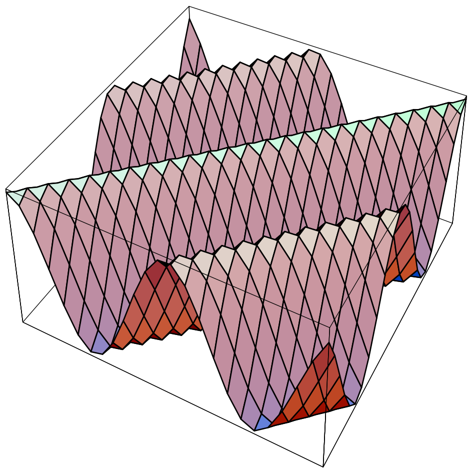

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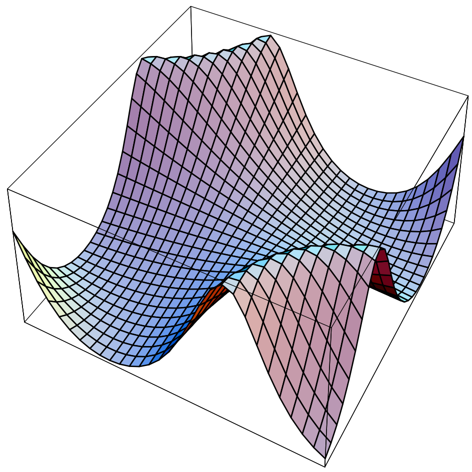

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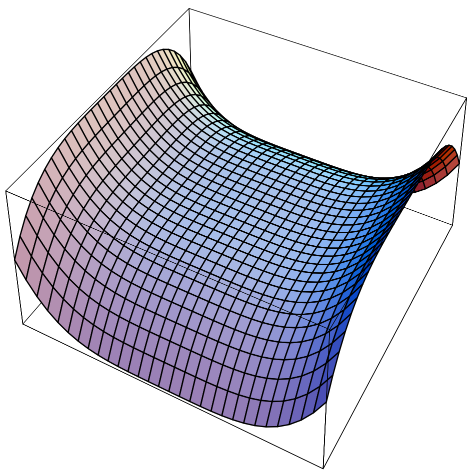

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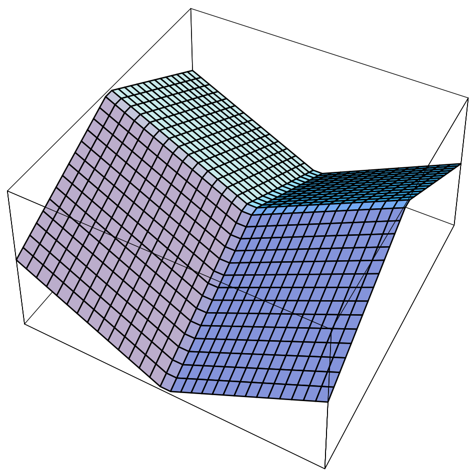


VI

Enter I,II,III,IV,V or VI here	Equation
	$z = \lvert x\rvert - \lvert y\rvert$
	$z = y^2 \log(x)$
	$z = \cos(5(x-y))$
	$z = x + 2y$
	$z = x/(2+\sin(xy))$
	$z = x^4 - y^4$

Solution:

Enter I,II,III,IV,V or VI here	Equation
VI	$z = \|x\| - \|y\|$
I	$z = y^2 \log(x)$
III	$z = \cos(5(x - y))$
II	$x + 2y$
IV	$z = x/(2 + \sin(xy))$
V	$z = x^4 - y^4$

Problem 4) (10 points)

Find the equation of the plane containing $A = (1, 1, 0)$ which is perpendicular to the planes

$$x + y + z = 1$$

and

$$x - y - z = 1 \ .$$

Solution:
The crossproduct of the vectors $\vec{v} = \langle 1, 1, 1\rangle$ and $\vec{w} = \langle 1, -1, -1\rangle$ is normal to the plane. Since $\vec{v} \times \vec{w} = \langle 0, 2, -2\rangle$, we have the equation $2y - 2z = d$ where d is a constant. Plggin in a point like $A = (1, 1, 0)$ gives $d = 2$. The equation of the plane is $y - z = 1$.

Problem 5) (10 points)

Find the distance between the point $P = (1, 0, -1)$ and the line which contains the points $A = (1, 1, 1)$ and $B = (0, 2, 1)$.
To do so:

a) (4 points) Find first a parametrization $\vec{r}(t) = A + t\vec{v}$ of the line.

b) (6 points) Now find the distance.

Solution:
a) $\vec{v} = \vec{AB} = ((0,2,1) - (1,1,1)) = (-1,1,0)$ is in the line. ' A parametrization is $\vec{r}(t) = (1,0,-1) + t(-1,1,0)$. We can write it as $\vec{r}(t) = (1-t, t, -1)$.
b) Use the distance formula: $|\vec{AP} \times \vec{v}|/|\vec{v}| = |(0,-1,-2) \times (-1,1,0)|/|(-1,1,0)| = \boxed{3/\sqrt{2}}$.

Problem 6) (10 points)

The angle between two planes is defined as the angle between the two normal vectors of the planes. Given the planes $x - z = 1$ and $y + z = 3$.

a) (2 point) Find the normal vector $\vec{v}$ to the first plane and the normal $\vec{w}$ to the second plane.

b) (2 points) Find the angle α between the two planes.

c) (2) points) Verify that $P = (4,0,3)$ and $Q = (1,3,0)$ are on the intersection of the two planes.

d) (2 points) Find a parametrization of the line of intersection of the two planes.

e) (2 points) Find the symmetric equation of the line of intersection of the two planes.

Solution:
a) $\vec{v} = (1,0,-1)$ and $\vec{w} = (0,1,1)$. b) $\cos(\alpha) = \vec{v} \cdot \vec{w}/(|\vec{v}||\vec{w}|) = -1/2$. So, $\alpha = 2\pi/3$.
c) Plug into $(x,y,z) = (4,0,3)$ and $(x,y,z) = (1,3,0)$ into the equations. d) $P + t(P - Q) = (4-3t, 0, 3-3t)$.
e) $(x-1)/1 = (y-2)/1 = -(z-2)/3$.

Problem 7) (10 points)

Find the implicit equation

$$ax + by + cz = d$$

of the plane which contains the line

$$\vec{r}(t) = P + t\vec{v} = (-1,1,1) + t(3,4,5)$$

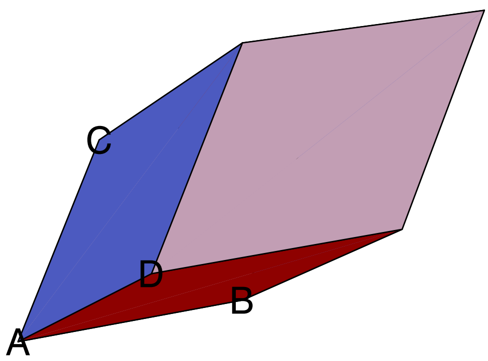
and the point $Q = (7,7,9)$.

Solution:
The vector $\vec{v} = (3,4,5)$ as well as the vector $\vec{PQ} = (8,6,8)$ are in the plane. Their cross product $\vec{v} \times \vec{PQ} = (2,16,-14)$ is perpendicular to the plane. The plane has therefore the equation $2x + 16y - 14z = d$. The constant $d = 0$ can be obtained by pluggin in a point. Fro example the point $(-1,1,1)$. The final answer is $2x + 16y - 14z = 0$.

Problem 8) (10 points)

a) (6 points) Find the surface area of the parallelepiped which contains the points $A = (0,0,0), B = (1,1,0), C = (0,1,1), D = (1,0,1)$.

b) (4 points) Find the volume of the solid.



Solution:
a) The surface of the parallelepiped has 6 faces. Each of the faces can be computed with a cross product. If $\vec{u} = \vec{AB} = (1,1,0)$, $\vec{v} = \vec{AC} = (0,1,1)$ and $\vec{w} = \vec{AD} = (1,0,1)$, then the area is $2|\vec{u} \times \vec{v}| + 2\vec{u} \times \vec{w} + 2\vec{v} \times \vec{w} = 2|(1,-1,1)||2|(1,-1,-1)| + 2|(1,1,-1)| = 2\sqrt{3} + 2\sqrt{3} + 2\sqrt{3} = \boxed{6\sqrt{3}}$.
b) The volume is the triple scalar product of $\vec{u}$, $\vec{v}$ and $\vec{w}$. It is $|(1,1,0) \cdot (1,1,-1)| = \boxed{2}$.

Problem 9) (10 points)

a) (4 points) We define a scalar valued function which has as argument two vectors:

$$f(\vec{v},\vec{w}) = \frac{|\vec{v}\times\vec{w}|}{\vec{v}\cdot\vec{w}}$$

What is $f((1,0,0),(1,1,0))$?

b) (6 points) The function $f(\vec{v},\vec{w})$ turns out to be a function of the angle α between $\vec{v}$ and $\vec{w}$ only. What is this function?

Solution:
a) The cross product is $(0,0,1)$. The dot product is 1. The result is $1/1 = \boxed{1}$.
b) Write $|\vec{v}\times\vec{w}| = |\vec{v}||\vec{w}|\sin(\alpha)$ and $\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos(\alpha)$. The quotient is $\sin(\alpha)/\cos(\alpha) = \boxed{\tan(\alpha)}$.

Problem 10) (10 points)

The set of points P for which the distance from P to $A = (1,2,3)$ is equal to the distance from P to $B = (5,8,5)$ forms a plane S.

a) Find the equation $ax + by + cz = d$ of the plane S.

b) Find the distance from A to S.

Solution:
a) The key insight is that the point $Q = (A+B)/2 = (3,5,4)$ is in the middle of the two points. The plane has to pass through this point. The normal vector is parallel to $\vec{n} = \langle 4,6,2\rangle$. The equation of the plane is

$$4x + 6y + 2z = 50\ .$$

b) The distance from A to S is half the distance from A to B which is $|\vec{AB}|/2 = |\langle 4,6,2\rangle|/2 = \sqrt{56}/2$.

Problem 11) (10 points)

Find the distance between the z-axis and the line

$$L: \frac{x+2}{4} = \frac{y-1}{3} = \frac{z}{2}.$$

Solution:
The z axes can be parametrized as

$$\vec{r}(t) = P + t\vec{v} = \langle 0,0,0 \rangle + t\langle 0,0,1 \rangle$$

The line L can be parametrized as

$$\vec{r}(t) = Q + t\vec{w} = \langle -2,1,0 \rangle + t\langle 4,3,2 \rangle$$

The distance is the length of the projection of $\vec{PQ} = \langle -2,1,0 \rangle$ onto the normal vector $\vec{n} = \vec{v} \times \vec{w} = \langle -3,4,0 \rangle$. This is

$$d = \frac{|\langle -2,1,0 \rangle \cdot \langle -3,4,0 \rangle|}{|\langle -3,4,0 \rangle|} = 10/5 = 2 \ .$$

The answer is 2.