Multivariable Calculus

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Abstract

This is an extended syllabus for this summer. It is actually telling the story of the entire course in a condensed form. These 8 pages can be a guide through the semester. The material is arranged in 6 chapters and delivered in the 6 weeks of the course. Each chapter has 4 sections, two sections for each day. While it make sense to read in a text book beside following the lectures, I want you to focus on the lectures. Textbooks have a lot of additional material and notation. Some of it can distract, other can even confuse. While a selected read in an other source can be helpful to get a second opinion and reconciliation of confusion and merging of different sources is an important aspect of learning, it can be sufficient and save time, to focus on the lectures delivered in class. It goes without saying that homework is extremely important. Mathematics can only be learned by solving problems.

Chapter 1. Geometry and Space

Section 1.1: Space, distance, geometrical objects

After an overview over the syllabus, we use **coordinates** like $P = (3, 4, 5)$ to describe points P in space. As promoted by **Descartes** in the 16'th century, geometry can be described algebraically when a **coordinate system** is introduced. A fundamental notion is the **distance** $d(P, Q) = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$ between two points $P = (x, y, z)$ and $Q = (a, b, c)$. This formula makes use of **Pythagoras** theorem. In order to get a feel about space, we look at some geometric objects defined by coordinates. We will focus on simple examples like **cylinders** and **spheres** and learn how to find the **center** and **radius** of a sphere given as a quadratic expression in x, y, z. This method is called the **completion of the square** and is based on one of the oldest techniques discovered in mathematics.

Section 1.2: Vectors, dot product, projections

Two points P, Q in space define a **vector** $\vec{PQ}$ at P. It has its **head** at Q and its **tail** at P. The vector connects the initial point P with the end point Q. Vectors can be attached everywhere in space, but they are identified if they have the same **length** and the same **direction**. Vectors can describe **velocities**, **forces** or **color** or **data**. The **components** of a vector $\vec{PQ}$ connecting a point $P = (a, b, c)$ with a point $Q = (x, y, z)$ are the entries of the vector $\langle x - a, y - b, z - c \rangle$.

Examples of vectors are the **zero vector** $\vec{0} = \langle 0, 0, 0 \rangle$, and the **standard basis vectors** $\vec{i} = \langle 1, 0, 0 \rangle, \vec{j} = \langle 0, 1, 0 \rangle, \vec{k} = \langle 0, 0, 1 \rangle$. We introduce **addition**, **subtraction** and **scalar multiplication** of vectors both geometrically as well as algebraically. The **dot product** $\vec{v} \cdot \vec{w}$ between two vectors, which results in a **scalar**, allows to compute **length**, **angles** and **projections**. By assuming the trigonometric cos-formula, we motivate the important **angle formula** $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos \alpha$. It allows to define and compute the **length** of a vector and the **angle** between two vectors with the help of a dot product.

Section 1.3: The cross product and triple scalar product

After a short review of the dot product we introduce the **cross product** $\vec{v} \times \vec{w}$ of two vectors $\vec{v} = \langle a, b, c \rangle$ and $\vec{w} = \langle p, q, r \rangle$ in space. This new vector $\langle br - cq, cp - ar, aq - bp \rangle$ is perpendicular to both vectors $\langle a, b, c \rangle$ and $\langle p, q, r \rangle$. The product can be valuable for many things: it is useful for example to compute **areas** of parallelograms, or the **distance** between a point and a line, to **construct** a plane through three points or to **intersect** two planes. We prove a formula $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$ for a quantity which can be interpreted geometrically as an **area** of the parallelepiped spanned by $\vec{v}$ and $\vec{w}$. Finally, we look at the **triple scalar product** $(\vec{u} \times \vec{v}) \cdot \vec{w}$ which is a scalar and is the **signed volume** of the parallelepiped spanned by $\vec{u}, \vec{v}$ and $\vec{w}$. Its sign tells about the **orientation** of the coordinate system defined by the three vectors. The triple scalar product is zero if and only if the three vectors are in a common plane.

Section 1.4: Lines, planes and distances

Because the $\langle a, b, c \rangle = \vec{n} = \vec{u} \times \vec{v}$ is perpendicular to $\vec{x} - \vec{w}$ if $\vec{x}, \vec{w}$ are both in the plane spanned by $\vec{u}$ and $\vec{v}$, we are led to the equation $ax + by + cz = d$ of the plane. Planes can be visualized by their **traces**, the intersection with coordinate planes as well as their **intercepts**, the intersection with coordinate axes. We often know the normal vector $\vec{n} = \langle a, b, c \rangle$ to a plane and can determine the constant d by plugging in a known point (x, y, z) on equation $ax + by + cz = d$. We introduce **lines** by the parameterization $\vec{r}(t) = \vec{OP} + t\vec{v}$, where P is a point on the line and $\vec{v} = \langle a, b, c \rangle$ is a vector telling the direction of the line. If $P = (o, p, q)$, then $(x - o)/a = (y - p)/b = (z - q)/c$ is called the **symmetric equation** of a line. It can be interpreted as the intersection of two planes. As an application of dot and cross products, we look at various **distance formulas**. Especially, we compute the distance from a point to a plane, the distance from a point to a line or the distance between two lines. We will also see how to compute distances between points, lines, planes, cylinders and spheres.

Chapter 2. Curves and Surfaces

Section 2.1: Functions, level surfaces, quadrics

We first focus on functions $f(x, y)$ of two variables. The **graph** of a function $f(x, y)$ of two variables is defined as the set of points (x, y, z) for which $z - f(x, y) = 0$. We look at a few examples and match some graphs with functions $f(x, y)$. **Traces**, the intersection of the graph with the coordinate planes as well as **generalized traces** like $f(x,y) = c$ which are called **level curves** of f help to visualize surfaces. The set of all level curves forms a so called **contour map**. After a short review of **conic sections** like **ellipses**, **parabola** and **hyperbola** in two dimensions, we look at more general surfaces of the form $g(x,y,z) = 0$. We start with known examples like the **sphere** and the **plane**. If $g(x,y,z)$ is a function which only involves linear and quadratic terms, the level surface is called a **quadric**. Important quadrics are **spheres**, **ellipsoids**, **cones**, **paraboloids**, **cylinders** as well as various **hyperboloids**.

Section 2.2: Parametric surfaces

Surfaces can be described in two fundamental ways: implicitly or parametrically. Implicit descriptions $g(x,y,z) = 0$ like $x^2 + y^2 + z^2 - 1 = 0$ have been introduced already earlier in the course. We look now at **parametrized surfaces** $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ like the **sphere** $\vec{r}(\theta,\phi) = \langle \rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi)\rangle$ where ρ is a fixed constant, and ϕ, θ are **Euler angles**. In many cases, it is possible to switch from a parametric surface to an implicit and back. Examples, where this works are the **plane**, **spheres**, **graphs of functions** and **surfaces of revolution**. Using a computer, one can **visualize** also complicated surfaces. Parametrization of surfaces is important in **geodesy**, where they appear as maps or in **computer generated imaging**, where the parameterization $\vec{r}(u,v)$ is called the **"uv-map"**. Parametrizations of surfaces make use of **cylindrical coordinates** (r,θ,z), where $r \geq 0$ is the distance to the z-axes and $0 \leq \theta < 2\pi$ is an angle. We also introducd **spherical coordinates** (ρ,θ,ϕ) where ρ is the distance to the origin and θ, ϕ are the Euler angles.

Section 2.3: Curves, velocity, acceleration

Curves are one-dimensional objects in the plane or in space. They take many different forms. We have already seen **level curves** of functions of two variables or **lines**. We define curves by parameterization $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, where the **parameter** t is given in some parameter interval $I = [a,b]$. A parametrization has more information than the curve itself. It is a dynamical and constructive description which also tells, how the curve is traced if t is interpreted a **time variable**. **Grid curves** of parametrized surfaces are other examples of curves. Differentiation of a parametrization $\vec{r}(t)$ leads to the **velocity** $\vec{r}\,'(t)$, a vector which is tangent to $\vec{r}(t)$. A second differentiation with respect to t gives the **acceleration** vector $\vec{r}\,''(t)$. The **speed** $|r'(t)|$ is a scalar. We also learn how to get from $\vec{r}''(t)$ and $\vec{r}\,'(0)$ and $\vec{r}(0)$ the position $\vec{r}(t)$ by integration. A nice application is the **free fall**, This is a case, where the acceleration vector of the curve is constant.

Section 2.4: Arc length and curvature

To derive the **arc length** of a curve, we first determine the length for a polygon and pass then to the limit, when the number of points of the polygon goes to infinity and the distance between neighboring points goes to zero. The polygon approaches so the given curve and leads to the one-dimensional **arc length** integral $\int_a^b |r'(t)| \; dt$. A re-parametrization of a curve does not change the arc length. We look at various examples. The **curvature** $\kappa(t)$ of a curve measures how much a curve is bent. Both acceleration and curvature involve second derivatives, but curvature is an intrinsic quantity of the curve which does not depend on the parameterization. One "feels" the acceleration and "sees" the curvature. We will see two formulas for curvature.

One of them is $\kappa(t) = |\vec{r}\,'(t) \times \vec{r}\,''(t)|/|r\,'(t)|$ and prove that they are equivalent. We introduce also the **unit normal vector** $\vec{T}$, the **normal vector** $\vec{N}$ as well as the **bi-normal vector** $\vec{B}$. These three vectors are perpendicular to each other.

Chapter 3. Linearization and Gradient

Section 3.1: Partial derivatives and partial differential equations

For functions $f(x)$ of one variable, continuity can fail in three different ways: with jump discontinuities, infinities or singular oscillations. In two dimensions, even more interesting things can happen but we only look at one or two examples which illustrates such **catastrophes**. From now on, we will assume all functions to be **smooth**: we can take as many derivatives in any variables as we wish. We introduce then **partial derivatives** $f_x = \partial_x f = \frac{\partial f}{\partial x}$ and derive **Clairot's theorem** $f_{xy} = f_{yx}$. This allows to compute f_{xxyx} for $f(x,y) = x^2\sin(\sin(9y)) + x\tan(y)$ very fast. To practice differentiation and to get a glimpse of how calculus is used in science, we check whether functions are solutions to **partial differential equations**, abbreviated **PDE's**. More precisely, we look at the **transport equation** $f_x(t,x) = f_t(t,x)$, the **wave equation** $f_{tt}(t,x) = f_{xx}(t,x)$ and the **heat equation** $f_t(t,x) = f_{xx}(t,x)$. Partial differential equations are extremely important in science. They form the fabric we are made of.

Section 3.2 Linear approximation and tangents

Linearization is an important concept in science because many physical laws are linearization of more complicated laws. Linearization is also useful to estimate things fast or find special points like intersections of curves or surfaces. After a review of linearization of functions of one variables, we introduce the **linearization** of a function $f(x,y)$ of two variables at a point (p,q). It is defined as the function $L(x,y) = f(p,q) + f_x(p,q)(x-p) + f_y(p,q)(y-q)$. The **tangent line** $ax + by = d$ at a point (p,q) is a level curve of L and $a = f_x, b = f_y$. Linearization works similarly in three dimensions, where it allows to compute the **tangent plane** $ax + by + cz = d$.

Section 3.3: Chain rule and implicit differentiation

The chain rule $d/dt f(g(t)) = f'(g(t))g'(t)$ in one dimension has generalizations in higher dimensions. In several dimensions, we have the chain rule $d/dt f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$, where $\nabla f = \langle f_x, f_y, f_z\rangle$ is the gradient. Written out, this is $d/dt f(x(t),y(t),z(t)) = f_x(x(t),y(t),z(t))x'(t) + f_y(x(t),y(t),z(t))y'(t) + f_z(x(t),y(t),z(t))z'(t)$. All other versions of chain rules you might see can be derived from this. A nice application of the chain rule is **implicit differentiation**: if $f(x,y,z) = 0$ defines a surface which looks locally like $z = g(x,y)$ and because $f_x + f_z z' = 0$ we can compute the partial derivatives g_x nd g_y of g without known g.

Section 3.4: Gradient and directional derivative

The **gradient** of a function is an important concept. It is a nice tool to describe the geometry of surfaces because the gradient of a function $f(x,y)$ is perpendicular to the level curve $f(x,y) = c$ passing through that point. Similarly, the gradient of a function $g(x,y,z)$ at a point is perpendicular to the level surface $f(x,y,z) = c$ through that point. One can see this both using linearization or by using the chain rule. for a curve $\vec{r}(t)$ on the surface $f(\vec{r}(t)) = 0$. A special case is $g(x,y,z) = ax + by + cz = d$, where $\nabla g = \langle a,b,c\rangle$. We review tangent planes and tangent lines. Now we introduce the **directional derivative** $D_v(f)$ as $D_v f = \nabla f \cdot v$ for unit vectors v. Partial derivatives are special directional derivatives. The direction of the normal vector gives a nonnegative partial derivative. Moving into the direction of the normal vector, increases f because $D_{\nabla f/||\nabla f} f = |\nabla f|$. In other words, the gradient vector points in the direction of **steepest ascent**.

Chapter 4. Extrema and Double integrals

Section 4.1: Extrema, second derivative test

A central application of multi-variable calculus is to **extremize** functions $f(x,y)$ of two variables. We first identify **critical points**, points where the gradient $\nabla f(x,y)$ vanishes. The nature of these critical points (p,q) can be established using the **second derivative test**. Let (p,q) be a critical point and let $D = f_{xx}f_{yy} - f_{xy}^2$ denote the **discriminant**. There will be three fundamentally different cases: **local maxima**, **local minima** as well as **saddle points**. If $D < 0$, then (p,q) is a saddle point, if $D > 0$ and $f_{xx} < 0$ then we have a local maximum, if $D > 0$ and $f_{xx} > 0$ then we have a local minimum. If $D = 0$, we can not determine the nature of the critical point from the second derivatives alone.

Section 4.2: Extrema with constraints

We extremize a function $f(x,y)$ in the presence of a **constraint** $g(x,y) = 0$. A necessary condition for a critical point is that the gradients of f and g are parallel. This leads to a system of equations which are called the **Lagrange equations** $\nabla f = \lambda \nabla g, g = 0$. When extremizing functions on a domain bounded by a curve $g(x,y) = 0$, we have to solve two problems: find the extrema in the interior and the extrema on the boundary. The second problem is a Lagrange problem. Lagrange problems can be considered in any dimension. We might look also at exampls of extremizing functions $f(x,y,z)$ of three variables, under the constrant $g(x,y) = 0, h(x,y) = 0$.

Section 4.3: Double integrals

Integration in two dimensions is first done on **rectangles**, then on regions bound by graphs of functions. Depending on whether graphs $y = c(x), y = d(x)$ or graphs $x = a(y), y = b(y)$ are the boundaries, we call the region **Type I** or **Type II**. Similar than in one dimension, there is a **Riemann sum approximation** of the integral. This allows us to derive results like **Fubini's theorem** on the change of the integration order. An application of double integration is the computation of **area**, for which $f(x,y) = 1$. We practice translating back and forth from a region to the integral. We also learn how to change of order of integration in regions which are both type I and type II. Sometimes, one can integrate one of the integrals only. A double integral can be interpreted as the signed volume under the graph of the function.

Section 4.4: Polar coordinates and surface area

Many planar regions can be described better in **polar coordinates** (r,θ), where r is the distance to the origin and θ is the angle with the positive x-axes. Examples of regions which can be treated like that are **polar region** is $r \leq g(\theta)$ which trace flower-like shapes in the plane. An other application of double integrals is **surface area**. We derive the formula $\int\int_R |r_u \times r_v|\ dudv$ and give examples like graphs, surfaces of revolution and especially the sphere. Similar as for arc length, it is easy to give examples, where the surface area can be computed in closed form, like triangles, parts of the sphere or cylinder or paraboloid. We will see also examples where one has to change the order of integration in order to find the surface area.

Chapter 5. Triple integrals and Line integrals

Section 5.1: Triple integrals

Triple integrals describe volumes, moment of inertias or centers of masses of solids. First introduced for cubes the triple integral is then extended to more general regions which have as boundaries graphs of functions of two variables. A major difficulty is the task to **set up a triple integral**. In most cases which appear in calculus text books, the region is contained between the graphs of two functions of two variables and if not, then this is an indication that a different coordinate system should be used. Applications make the topic of integration more interesting. It also allows to practice how to set up integrals. Applications are computations of **mass** $\int\int\int_E \delta(x,y,z)\ dxdydz$, **moment of inertia** $\int\int\int_E (x^2+y^2+z^2)\ dxdydz$, **center of mass**, $\int\int\int \langle x,y,z\rangle\ dV$ the **expectation** $\mathrm{E}[X] = \int\int\int X(x,y,z)\ dV / \int\int\int\ dV$ of a random variable $X(x,y,z)$ on a region Ω.

Section 5.2: Spherical and cylindrical coordinates

Some regions in space can be described better in **cylindrical coordinates** (r,θ,z) which are just polar coordinates for the x,y variables in space. Examples of such regions are parts of cylinders or solids of revolution. The important factor to include when changing to cylindrical coordinates is r. Other regions are integrated over better in **spherical coordinates** (ρ,ϕ,θ). Example of such regions are parts of cones or spheres. The important factor to include when changing to spherical coordinates is $\rho^2 \sin(\phi)$. As an application we compute **moments of inertia** of some bodies.

Section 5.3: Vector fields and line integrals

Vector fields occur as force fields or velocity fields or mechanics. They can be defined in 2 or 3 dimensions. We learn how to **match vector fields** with formulas describing them and introduce **flow lines**, parametrized curves $\vec{r}(t)$ for which $\vec{F}(\vec{r}(t))$ is parallel to $\vec{r}\,'(t)$ at all times. Given a parametrized curve $\vec{r}(t)$ and a vector field $\vec{F}$, we can define the **line integral** $\int_C F(\vec{r}(t))\ \vec{r}\,'(t)\ dt$ along a curve in the presence of a vector field. An important example is if $\vec{F}$ is a **force field** and where line integral is **work**.

Section 5.4: Fundamental theorem of line integrals

For a class of vector fields which we call **conservative vector fields** one can evaluate the line integral using the **fundamental theorem of line integrals**. We distinguish first between **path independent fields** which are also called **conservative** and **gradient fields** where $\vec{F} = \nabla f$ and **irrotational fields**, for which $\text{curl}(F) = Q_x - P_y = 0$. While the first two properties are always equivalent, the later property is equivalent only under the condition that the region in which $\vec{F}$ is defined is simply connected. A region is **simply connected** if every every closed path in the region can be contracted to a point. The plane with the unit disc cut out is not simply connected because the path $\langle 2\cos(t), 2\sin(t)\rangle$ can not be pulled together to a point. In two dimensions, the curl of a field $\text{curl}\langle P, Q\rangle = Q_x - P_y$ measures the **vorticity** of the field.

Chapter 6. Vector fields and Integral theorems

Section 6.1: Green theorem and Curl

Greens theorem relates a line integral along a closed curve C with a double integral of a derivative of the vector field in the region G enclosed by the curve: $\int\int_G \text{curl}(\vec{F})(x,y)\ dxdy = \int_C \vec{F}(\vec{r}(t))\ \vec{r}\,'(t)\ dt$. The theorem is useful for example to **compute areas**: just take a field $\vec{F} = \langle 0, x\rangle$ which has constant curl 1. It also allows an easy computation of line integrals in certain cases. Greens theorem implies that if $\text{curl}(F) = 0$ everywhere in the plane, then the field has the **closed loop property** and is therefore conservative. The **curl** of a vector field $\vec{F} = \langle P, Q, R\rangle$ in three dimensions is a new vector field which can be computed as $\nabla \times \vec{F}$. The three components of $\text{curl}(F)$ are the vorticity of the vector field in the x,y and z direction.

Section 6.2: Flux integrals and Derivative overview

Given a surface membrane S and a fluid with has velocity $\vec{F}(x,y,z)$ at the point (x,y,z). the amount of fluid which passes through the membrane in unit time is the **flux integral** $\int\int_S \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\ dudv$ of a vector field $\vec{F}$ through a surface S. Obviously, the angle between $\vec{F}$ and the normal vector $\vec{n} = \vec{r}_u \times \vec{r}_v$ determines the sign of $\vec{dS} = \vec{F}\cdot\vec{n}dudv$. A lot of things come together in this definition: the parametrization of surfaces, the dot and cross product, as well as double integrals. We discuss how the derivatives **div**, **grad** and **curl** all fit together. In one dimensions, there is only one derivative, in two dimensions, there are two derivatives grad and curl and in three dimensions, there are three derivatives grad, , curl and div.

Section 6.3: Stokes theorem and Maxwell

Stokes theorem looks as if Greens theorem was lifted into three dimensions and where the region is replaced by a surface. Stokes theorem tells that one can replace the line integral along the boundary C of the surface by an integral of the "curl" of the field over the surface: $\int_C \vec{F}\ dr = \int\int_S \text{curl}(F)\dot{d}S$. As in Greens theorem, it is important to have the right orientations of the objects. We see different examples on how to use Stokes theorem to compute line integrals or to compute flux integrals. There are many applications of Stokes theorem. The theorem sheds light onto the equations of electromagnetism which describe light. Stokes theorem illustrates why the line integral of an irrotational field along a closed curve in space is zero: it is the flux of the curl of $\vec{F}$ through a surface S bound by the curve C.

Section 6.4 Divergence theorem and Summary

The **divergence** of a vector field $\vec{F} = \langle P, Q, R\rangle$ inside solid E is related to the flux of the vector field $\vec{F}$ through the boundary S of the solid. This correspondence is given by the **divergence theorem**. It equates the "local expansion rate" integrated over the solid $\int\int\int_E \text{div}(\vec{F})\ dV$ of a vector field $\vec{F}$ with the flux $\int\int_S \vec{F}\cdot\vec{dS}$ through the boundary surface S of E. It is useful for example to compute the **gravitational field** inside a solid like the sphere or a hollow sphere. The divergence theorem tells what divergence is: a measure of how much the field expands at a point. Finally, we give an overview over all integral theorems. In one dimension, there is one integral theorem the **fundamental theorem of calculus**. In two dimensions, there are two integral theorems, **the fundamental theorem of line integrals in the plane** as well as **Greens theorem**. In three dimensions there are three integral theorems: the **fundamental theorem of line integrals in space**, **Stokes theorem** and the **divergence theorem**. All these integral theorems are of the form $\int_{\delta G} F = \int_G dF$, where δG is the boundary of an object G and dF is the derivative of the field F.

In this last lecture we also will collect the Mathematica project which allows you to see all the material from a more computational point of view.