

## Homework for Chapter 6. Integral Theorems

### Section 6.1:

- 1) (Greens theorem) Calculate the line integral  $\int_C \vec{F} \cdot d\vec{r}$  with  $\vec{F} = \langle 2y + x \sin(y), x^2 \cos(y) - 3y^{200} \sin(y) \rangle$  along a triangle  $C$  with edges  $(0, 0)$ ,  $(1, 0)$  and  $(1, 1)$ .

**Solution:**

$\text{curl}(F)(x, y) = 2x \cos(y) - 2 - x \cos(y) = x \cos(y) - 2$ . By Green's theorem, we have to integrate this function over the region  $R$  enclosed by the triangle:

$$\int_0^1 \int_0^x x \cos(y) - 2 \, dy dx = -1 - \cos(1) + \sin(1).$$

- 2) (Greens theorem) Evaluate the line integral of the vector field  $\vec{F}(x, y) = \langle xy^2, x^2 \rangle$  along the rectangle with vertices  $(0, 0)$ ,  $(2, 0)$ ,  $(2, 3)$ ,  $(0, 3)$ .

**Solution:**

Integrating  $\text{curl}(F) = 2x - 2xy$  over the rectangle gives

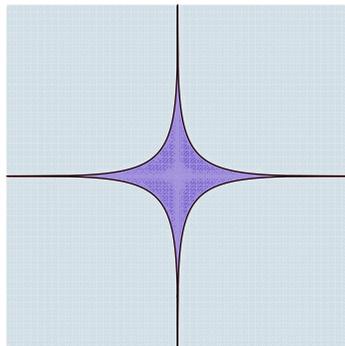
$$\int_0^2 \int_0^3 2x - 2xy \, dy dx = \int_0^2 6x - 9x \, dx = -12/2 = -6$$

The line integral is a sum of four line integrals:  $\int_0^2 \langle t0^2, t^2 \rangle \cdot \langle 1, 0 \rangle \, dt = 0$  and  $\int_0^3 \langle 2t^2, 4 \rangle \cdot \langle 0, 1 \rangle \, dt = 12$  and  $-\int_0^2 \langle 9t, t^2 \rangle \cdot \langle 1, 0 \rangle \, dt = -18$  as well as  $\int_0^3 \langle 0t^2, 0 \rangle \cdot \langle 0, 1 \rangle \, dt = 0$ . The sum is also  $-6$ .

- 3) (Greens theorem) Find the area of the region bounded by the **hypocycloid**

$$\vec{r}(t) = \langle \cos^3(t), \sin^3(t) \rangle$$

using Green's theorem. The curve is parameterized by  $t \in [0, 2\pi]$ .



**Solution:**

Take a vector field  $\vec{F}(x, y) = \langle 0, x \rangle$  which has the curl 1. Then by Green the area is the line integral  $\int_0^{2\pi} \langle 0, \cos^3(t) \rangle \cdot \langle -3 \cos^2(t) \sin(t), 3 \sin^2(t) \cos(t) \rangle \, dt = 3 \int_0^{2\pi} \cos^4(t) \sin^2(t) \, dt = 3 \int_0^{2\pi} \sin^2(2t)/4(\cos(2t) + 1)/2 \, dt = 3/8\pi$ .

- 4) Let  $G$  be the region  $x^6 + y^6 \leq 1$ . Compute the line integral of the vector field  $\vec{F}(x, y) = \langle x^6, y^6 \rangle$  along the boundary.

**Solution:**

The curl of  $\vec{F}$  is zero. Therefore the line integral is zero by Greens theorem. Alternatively, we could invoke the fundamental theorem of line integrals.

- 5) Let  $\vec{F}(x, y) = \langle -y/(x^2 + y^2), x/(x^2 + y^2) \rangle$ . Let  $C : \vec{r}(t) = \langle \cos(t), \sin(t) \rangle, t \in [0, 2\pi]$ .

a) Compute  $\int_C \vec{F} \cdot d\vec{r}$ .

b) Show that  $\text{curl}(\vec{F}) = 0$  everywhere for  $(x, y) \neq (0, 0)$ .

c) Let  $f(x, y) = \arctan(y/x)$ . Verify that  $\nabla f = \vec{F}$ .

d) Why do a) and b) not contradict the fact that a gradient field has the closed loop property? Why does a) and b) not contradict Green's theorem?

**Solution:**

a) The line integral is  $2\pi$ .

b) Direct differentiation gives  $Q_x(x, y) - P_y(x, y) = 0$ .

c) Use  $\arctan'(x) = 1/(1 + x^2)$ .

d) The function  $f(x, y)$  is not continuous everywhere. Also the vector field  $\vec{F}(x, y)$  is not smooth everywhere. There is a singularity at  $(0, 0)$ .

Greens theorem can not be involved because the vector field is not defined everywhere inside the disc.

## Section 6.2: Curl, Div and Flux

- 1) (Curl and Div) Find nonzero vector fields  $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$  for each of the following cases:
- $\vec{F}$  is irrotational but not incompressible.
  - $\vec{F}$  is incompressible but not irrotational.
  - $\vec{F}$  is irrotational and incompressible.
  - $\vec{F}$  is not irrotational and not incompressible.



The terminology in this problem comes from fluid dynamics where fluids can be incompressible, irrotational.

### Solution:

- $\langle x^2, y^2 \rangle$
- $\langle y^2, -x^2 \rangle$
- $\langle y, x \rangle$
- $\langle x^2 + y^3, x^3 - y^3 \rangle$

- 2) (Curl) The vector field  $\vec{F}(x, y, z) = \langle x, y, -2z \rangle$  satisfies  $\text{div}(\vec{F}) = 0$ . Can you find a vector field  $\vec{G}(x, y, z)$  such that  $\text{curl}(\vec{G}) = \vec{F}$ ? Such a field  $\vec{G}$  is called a **vector potential**.  
Hint. Write  $\vec{F}$  as a sum  $\langle x, 0, -z \rangle + \langle 0, y, -z \rangle$  and find vector potentials for each of the summand using a field you have seen in class.
- 3) (Flux integral) Evaluate the flux integral  $\int \int_S \langle 0, 0, yz \rangle \cdot d\vec{S}$ , where  $S$  is the surface with parametric equation  $x = uv, y = u + v, z = u - v$  on  $R : u^2 + v^2 \leq 1$ .

### Solution:

$\vec{r}_u = (v, 1, 1), \vec{r}_v = (u, 1, -1)$  so that  $\vec{r}_u \times \vec{r}_v = (-2, u + v, -u + v)$ . The flux integral is  $\iint_R \langle 0, 0, u^2 - v^2 \rangle \cdot \langle -2, u + v, -u + v \rangle dudv = \iint_R v^2u - u^3 - v^3 + u^2v dudv$  which is best evaluated using polar coordinates:  $\int_0^1 \int_0^{2\pi} r^4(\sin^2(\theta)\cos(\theta) - \cos^3(\theta) - \sin^3(\theta) + \cos^2(\theta)\sin(\theta)) d\theta dr = 0$ .

- 4) (Flux integral) Evaluate the flux integral  $\iint_S \text{curl}(F) \cdot d\vec{S}$  for  $\vec{F}(x, y, z) = \langle xy, yz, zx \rangle$ , where  $S$  is the part of the paraboloid  $z = 4 - x^2 - y^2$  that lies above the square  $[1, 0] \times [0, 1]$  and has an upward orientation.

### Solution:

$\text{curl}(F) = \langle -y, -z, -x \rangle$ . The parametrization  $\vec{r}(u, v) = \langle u, v, 4 - u^2 - v^2 \rangle$  gives  $r_u \times r_v = \langle 2u, 2v, 1 \rangle$  and  $\text{curl}(F)(\vec{r}(u, v)) = \langle -v, u^2 + v^2 - 4, -u \rangle$ . The flux integral is  $\int_0^1 \int_0^1 \langle -2uv + 2v(u^2 + v^2 - 4) - u \rangle dudv = -1/2 + 1/3 + 1/2 - 4 - 1/2 = -25/6$ .

- 5) (Flux integral) What is the relation between the flux of the vector field  $\vec{F} = \nabla g / |\nabla g|$  through the surface  $S : \{g = 1\}$  with  $g(x, y, z) = x^6 + y^4 + 2z^8$  and the surface area of  $S$ ?

### Solution:

The surface area is equal to the flux.

### Section 6.3: Stokes theorem

- 1) (Stokes theorem) Find  $\int_C \vec{F} \cdot d\vec{r}$ , where  $\vec{F}(x, y, z) = \langle x^2y, x^3/3, xy \rangle$  and  $C$  is the curve of intersection of the hyperbolic paraboloid  $z = y^2 - x^2$  and the cylinder  $x^2 + y^2 = 1$ , oriented counterclockwise as viewed from above.

**Solution:**

1. Solution: The curl of  $F$  is  $\text{curl}(F) = (x, -y, 0)$ . We can parametrize the hyperbolic paraboloid as  $\vec{r}(u, v) = (u \cos(v), u \sin(v), -u^2 \cos(2v))$ .  $\vec{r}_u \times \vec{r}_v = \langle 2u^2 \cos(v), -2u^2 \sin(v), u \rangle$ .  $\vec{F}(\vec{r}(u, v)) = \langle u \cos(v), -u \sin(v), 0 \rangle$ .  $\vec{F}(\vec{r}(u, v)) \cdot \vec{r}_u \times \vec{r}_v = 2u^3$ .

$$\int_0^1 \int_0^{2\pi} -2r^3 d\theta dr = \pi .$$

2. Solution: with the parametrization  $r(u, v) = (u, v, v^2 - u^2)$ , we have  $r_u \times r_v = \langle 2u, -2v, 1 \rangle$  and  $F(r(u, v)) = \langle u, -v, 0 \rangle$  so that  $F(r(u, v)) \cdot (r_u \times r_v) = 2u^2 + 2v^2$ . Integrating this over the disc gives  $\boxed{\pi}$ .

- 2) (Stokes theorem) If  $S$  is the surface  $x^6 + y^6 + z^6 = 1$  and assume  $\vec{F}$  is a smooth vector field. Explain why  $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = 0$ .

**Solution:**

The flux of  $\text{curl}(F)$  through a closed surface is zero by Stokes theorem and the fact that the surface does not have a boundary.

One can see this also by cutting the surface in two pieces and apply Stokes to both pieces.

- 3) (Stokes theorem) Evaluate the flux integral

$$\int \int_S \text{curl}(\vec{F}) \cdot d\vec{S} ,$$

where  $\vec{F}(x, y, z) = \langle xe^{yz}, x^2 + 2xyze^{xz}, x + z^2e^{xz}, ye^{xz} + ze^{xz} \rangle$  and where  $S$  is the part of the ellipsoid  $x^2 + y^2/4 + (z + 1)^2 = 2, z > 0$  oriented so that the normal vector points upwards.

**Solution:**

Stokes theorem assures that the flux integral is equal to the line integral along the boundary of the surface. The boundary is the ellipse  $\vec{r}(t) = \langle \cos(t), 2 \sin(t), 0 \rangle, 0 \leq t \leq 2\pi$ . The vector field on the  $xy$ -plane  $z = 0$  is

$$\vec{F}(x, y, 0) = \langle 0, x, ye^{x^2} \rangle .$$

To compute the line integral of this vector field along the boundary curve, compute  $\vec{r}'(t) = \langle -\sin(t), 2 \cos(t), 0 \rangle$  and  $\vec{F}(\vec{r}(t)) = \langle 0, \cos(t), 2 \sin(t)e^{\sin^2(t)} \rangle$ . The dot product of these two vectors is the function  $2 \cos^2(t)$ , the power. Integrating this over  $[0, 2\pi]$  gives  $\boxed{2\pi}$ .

- 4) (Stokes theorem) Find the line integral  $\int_C \vec{F} \cdot d\vec{r}$ , where  $C$  is the circle of radius 3 in the  $xz$ -plane oriented counter clockwise when looking from the point  $(0, 1, 0)$  onto the plane and where  $\vec{F}$  is the vector field

$$\vec{F}(x, y, z) = \langle 2x^2z + x^5, \cos(e^y), -2xz^2 + \sin(\sin(z)) \rangle .$$

**Hint.** Use a convenient surface  $S$  which has  $C$  as a boundary.

**Solution:**

The line integral can not be computed directly. The curl of  $\vec{F}$  is  $\langle 0, 2x^2 + 2z^2, 0 \rangle$ .

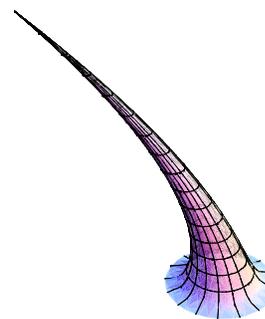
- 5) (Stokes theorem) Find the flux integral  $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$ , where

$$\vec{F}(x, y, z) = \langle 2 \cos(\pi y)e^{2x} + z^2, x^2 \cos(z\pi/2) - \pi \sin(\pi y)e^{2x}, 2xz \rangle$$

and  $S$  is the **thorn** surface parametrized by

$$\vec{r}(s, t) = \langle (1 - s^{1/3}) \cos(t) - 4s^2, (1 - s^{1/3}) \sin(t), 5s \rangle$$

with  $0 \leq t \leq 2\pi, 0 \leq s \leq 1$  and oriented so that the normal vectors point to the outside of the thorn.



**Solution:**

This problem can be solved in three different ways. 1. solution. The vector field  $\vec{F}$  is the sum of the gradient of  $f(x, y, z) = \cos(\pi y)e^{2x} + z^2x$  and  $\vec{G}(x, y, z) = \langle 0, x^2 \cos(z\pi/2), 0 \rangle$ . By Stokes theorem, the flux of  $\text{curl}(F) = \pi x^2 \sin(\pi z)/2, 0, 2x \cos(\pi z/2)$  is the line integral of  $\langle 0, x^2 \cos(z\pi/2), 0 \rangle$  along the boundary curve  $\vec{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$  which is  $\int_0^{2\pi} \langle 0, \cos^2(t), 0 \rangle \cdot \langle \sin(t), \cos(t), 0 \rangle dt = 0$ .

2. Solution. The flux is by Stokes theorem the line integral along the boundary  $\vec{r}(t)$  which is by Stokes theorem the flux integral of  $\text{curl}(\vec{F}) = \pi x^2 \sin(\pi z)/2, 0, 2x \cos(\pi z/2)$  through the disc with that boundary. This flux integral is zero because on the disc  $\text{curl}(\vec{F}(x, y, z)) = \langle 0, 0, 2x \rangle$  so that the flux is the double integral of  $2x$  over the disc which is zero.

3. Solution. The flux through the "thorn" together with the flux through the bottom disc (oriented downwards) closing the surface is zero because  $\text{div}(\text{curl}(\vec{F})) = 0$ . Therefore, the flux through the thorn is the same as the flux through the disc (oriented upwards) which is zero as in the 2. Solution. The result is again  $\boxed{0}$ .

### Section 6.4:

- 1) (Divergence theorem) Compute using the divergence theorem the flux of the vector field  $\vec{F}(x, y, z) = \langle 3y, xy, 2yz \rangle$  through the unit cube  $[0, 1] \times [0, 1] \times [0, 1]$ .

**Solution:**

The divergence of  $F$  is  $x + 2y$ . Integrating this over the unit cube gives  $1/2 + 1 = 3/2$ .

Optional computation: The cube consists of 6 faces. The flux through the face  $x = 1$  is  $\int_0^1 \int_0^1 3y \, dydz = 3/2$ .

The flux through the face  $x = 0$  is  $-3/2$ .

The flux through the face  $y = 1$  is  $\int_0^1 \int_0^1 x \, dx dz = 1/2$ .

The flux through the face  $y = 0$  is 0.

The flux through the face  $z = 1$  is  $\int_0^1 \int_0^1 2y \, dx dy = 1$ .

The flux through the face  $z = 0$  is  $\int_0^1 \int_0^1 0 \, dx dy = 0$ .

The sum of all these fluxes is  $3/2$ .

- 2) (Divergence theorem) Find the flux of the vector field  $\vec{F}(x, y, z) = \langle xy, yz, zx \rangle$  through the solid cylinder  $x^2 + y^2 \leq 1, 0 \leq z \leq 1$ .

**Solution:**

The divergence is  $(y + z + x)$ . Integrated over the cylinder gives  $\int_0^1 z \, dz \pi = \pi/2$ .

Optional: The flux of the vector field through the bottom is 0 because there the vector field has the form  $(*, *, 0)$  and the normal vector is  $(0, 0, -1)$ . The flux integral over the top is  $\iint_R x \, dx dy$  where  $R$  is the unit disc, which is zero. To compute the flux integral over the boundary of the cylinder, parametrize the cylinder as  $\vec{r}(\theta, z) = (\cos(\theta), \sin(\theta), z)$ . We have  $F(\vec{r}(u, v)) = (\cos(\theta)\sin(\theta), \sin(\theta)z, \cos(\theta)z)$  and  $r_u \times r_v = (\cos(\theta), \sin(\theta), 0)$ . The flux integral is  $\int_0^1 \int_0^{2\pi} \cos^2(\theta)\sin(\theta) + \sin^2(\theta)z \, d\theta dz = \pi/2$ .

- 3) (Divergence theorem) Use the divergence theorem to calculate the flux of  $\vec{F}(x, y, z) = \langle x^3, y^3, z^3 \rangle$  through the sphere  $S : x^2 + y^2 + z^2 = 1$  where the sphere is oriented so that the normal vector points outwards.

**Solution:**

The divergence of  $\vec{F}$  is  $3x^2 + 3y^2 + 3z^2 = 3\rho^2$ . By the divergence theorem, we have to integrate this over the interior of the sphere. This is done best in spherical coordinates.

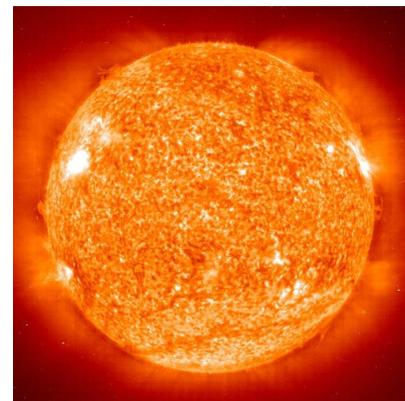
The integral is

$$\int_0^{2\pi} \int_0^\pi 3\rho^4 \sin(\phi) \, d\rho d\phi d\theta = 12\pi/5.$$

- 4) (Divergence theorem) Assume the vector field

$$\vec{F}(x, y, z) = \langle 5x^3 + 12xy^2, y^3 + e^y \sin(z), 5z^3 + e^y \cos(z) \rangle$$

is the magnetic field of the **sun** whose surface is a sphere of radius 3 oriented with the outward orientation. Compute the magnetic flux  $\iint_S \vec{F} \cdot d\vec{S}$ .

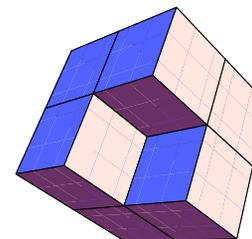
**Solution:**

The divergence is  $15x^2 + 15y^2 + 15z^2$ . We integrate this over the sphere to get by the divergence theorem the flux through the surface. To compute the triple integral we use spherical coordinates

$$\int_0^{2\pi} \int_0^\pi \int_0^1 15\rho^2 \rho^2 \sin(\phi) \, d\rho d\phi d\theta.$$

The result is  $15(3^5/5)4\pi = 4 \cdot 3^6 \cdot \pi = 2916\pi$ .

- 5) (Divergence theorem) Find  $\int \int_S \vec{F} \cdot d\vec{S}$ , where  $\vec{F}(x, y, z) = \langle x, y, z \rangle$  and  $S$  is the outwardly oriented surface obtained by removing the cube  $[1, 2] \times [1, 2] \times [1, 2]$  from the cube  $[0, 2] \times [0, 2] \times [0, 2]$ .

**Solution:**

The divergence is 3. By the divergence theorem, the result is 3 times the area of the solid  $E$  which is  $(8 - 1)3 = 21$ .