Oliver Knill, Harvard Summer School, Summer 2009

Homework for Chapter 5. Triple integrals and line integrals

Section 5.1: Triple integrals

1) (triple integrals) Evaluate the triple integral

$$\int_0^1 \int_0^z \int_0^{2y} z e^{-y^2} \, dxdydz \, .$$

Solution:
$\int_0^{2y} z e^{-y^2} \, dx = 2yze^{-y^2}$.
$\int_0^z 2yze^{-y^2} \, dy = z - ze^{-z^2}$.
$\int_0^1 z - ze^{-z^2} \, dz = z^2/2 - e^{-z^2}/2|_0^1 = \boxed{1/(2e)}$.

2) (triple integrals polar coordinates) Find the volume of the solid bounded by the paraboloids $z = x^2 + y^2$ and $z = 9 - (x^2 + y^2)$ and satisfying $x \geq 0$.

Solution:
Use cylindrical coordinates: The cylinders intersect when $r^2 = 9 - r^2$ or $r = \sqrt{9/2}$. We compute

$$\int_0^{\sqrt{9/2}} \int_{-\pi/2}^{\pi/2} (9 - r^2 - r^2) r \, d\theta dr = 81\pi/8 \, .$$

3) (triple integrals, polar coordinates) Find the **moment of inertia** $\int \int \int_E (x^2 + y^2) \, dV$ of a cone

$$E = \{x^2 + y^2 \leq z^2 \; 0 \leq z \leq 1 \} \, ,$$

if the cone has the z-axis as its center of symmetry.

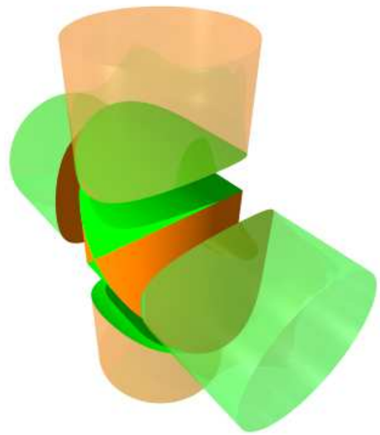
Solution:
$2\pi \int_0^1 \int_0^z r^3 \, drdz = 2\pi \int_0^1 z^4/4 \, dz = \boxed{\pi/10}$.

4) (triple integrals) Integrate $f(x, y, z) = x^2 + y^2 - z$ over the tetrahedron with vertices $(0, 0, 0), (1, 1, 0), (0, 1, 0), (0, 0, 3)$.

Solution:
Make a picture. The tetrahedron sits above a base which is a triangle in the xy plane. The roof is the plane $z = 3 - 3y$. The integral is

$$\int_0^1 \int_x^1 \int_0^{3-3y} x^2 + y^2 - z \, dzdydx = -7/40 \, .$$

5) (triple integral) This is an old classic problem: What is the volume of the body obtained by intersecting the solid cylinders $x^2 + z^2 \leq 1$ and $y^2 + z^2 \leq 1$?



Solution:
Integrate over 1/16 th, which has a triangle in the base. This leads to the integral

$$\int_0^1 \int_0^x \int_0^{\sqrt{1-x^2}} dzdydx = \int_0^1 x\sqrt{1 - x^2} \, dx = 1/3 \, .$$

The final result is 16/3.

Section 5.2: Spherical and cylindrical coordinates

1) (cylindrical or spherical coordinates?) The density of a solid $E = x^2 + y^2 - z^2 < 1, -1 < z < 1$. is given by the forth power of the distance to the z-axes: $\sigma(x, y, z) = (x^2 + y^2)^2$. Find its mass

$$M = \int\int\int_E (x^2 + y^2)^2 \, dxdydz \ .$$

Solution:
Use cylindrical coordinates:

$$\int_0^{2\pi} \int_{-1}^{1} \int_0^{\sqrt{1+z^2}} r^5 \, drdzd\theta = 2\pi \int_{-1}^{1} (1 + z^2)^6/6 \, dz = 2\pi 33472/9009 \ .$$

2) (cylindrical or spherical coordinates?) Find the moment of inertia $\int\int\int_E (x^2 + y^2) \, dV$ of the body E whose volume is given by the integral

$$\text{Vol}(E) = \int_0^{\pi/4} \int_0^{\pi/2} \int_0^{3} \rho^2 \sin(\phi) \, d\rho d\theta d\phi \ .$$

Solution:
This is a quarter of an "ice cream cone".
The moment of inertia integral integral is

$$\int_0^{\pi/2} \int_0^{3} \int_0^{\pi/4} \rho^2 \sin^2(\phi) \rho^2 \sin(\phi) \, d\phi d\rho d\theta$$

which is $\pi/2(3^5/5) \int_0^{\pi/4} \sin^3(\phi) \, d\phi = \pi/2(3^5/5)(2/3) = 81Pi/5$.

3) (integration in spherical coordinates?) A solid is described in spherical coordinates by the inequality $\rho \leq \sin(\phi)$. Find its volume.

Solution:
$2\pi \int_0^{\pi} \sin(\phi)^4/3 d\phi$.

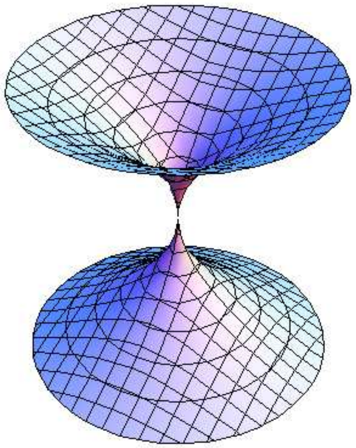
4) (cylindrical or spherical coordinates?) Integrate the function

$$f(x, y, z) = e^{(x^2+y^2+z^2)^{3/2}}$$

over the solid which lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$, which is in the first octant and which is above the cone $x^2 + y^2 = z^2$.

Solution:
Because we are in the first octant, $\theta \in [0, \pi/2]$ The ϕ angle is between 0 and $\pi/4$, the radius varies between 1 and 2.
$\int_1^2 \int_0^{\pi/2} \int_0^{\pi/4} e^{\rho^3} \rho^2 \sin(\phi) \, d\phi d\theta d\rho = (\sqrt{2}/2)(\pi/2) e^{\rho^3}/3|_1^2 = \boxed{(\sqrt{2}\pi/4)(e^8 - e)/3}$

5) (cylindrical or spherical coordinates?) Find the volume of the solid $x^2 + y^2 \leq z^4, z^2 \leq 1$.



Solution:
In cylindrical coordinates, the body is $r^2 \leq z^4$. This is a surface of revolution. We can integrate

$$\int_{-1}^{1} \int_0^{2\pi} \int_0^{z^2} r \, drd\theta dz$$

and get π.

Section 5.3: Vector fields and line integrals

1) (vector fields) The vector field $\vec{F}(x,y) = \langle x/r^3, y/r^3 \rangle$ appears in electrostatics, where $r = \sqrt{x^2+y^2}$ is the distance to the charge. Find a function $f(x,y)$ such that $\vec{F} = \nabla f$.

Solution:
The function is $f(x,y) = -1/\sqrt{x^2+y^2}$.

2) (vector fields) a) Draw the gradient vector field of the function $f(x,y) = \sin(x+y)$.
b) Draw the gradient vector field of the function $f(x,y) = (x-1)^2 + (y-2)^2$.
Hint: In both cases, draw first a contour map of f and use a property of gradients to draw the vector field $F(x,y) = \nabla f$.

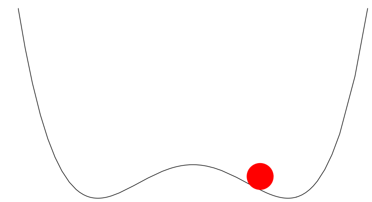
Solution:
a) The level curves of $f(x,y)$ are the same then the level curves of the function $g(x,y) = x+y$ because $\sin(x+y) = c$ means $x+y = \arcsin(c) = C$. The level curves are therefore straight lines.
$\nabla f(x,y) = (\cos(x+y), \cos(x+y))$ is a vector field which is perpendicular to the level curves. It vanishes at places, where $\sin(x+y) = 1$ or $\sin(x+y) = -1$.
b) The contour map consists of circles centered at $(1,2)$. The vector field is perpendicular to those circles.

3) (vector fields)
a) Is the vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle = \langle xy, x^2 \rangle$ a gradient field?
b) Is the vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle = \langle \sin(x)+y, \cos(y)+x \rangle$ a gradient field?
In both cases, give the potential if it exists and if there is no gradient, give a reason, why it is not a gradient field.

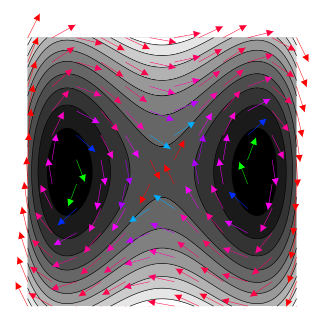
Solution:
a) No, because $Q_x = 2x$ and $P_y = x$, this can not be a gradient field $\langle P, Q \rangle = \langle f_x, f_y \rangle$.
b) Yes, the function is $f(x,y) = -\cos(x) + \sin(y) + xy$.

4) (vector fields) A ball rolls on the graph of the function $f(x) = x^4 - x^2$. Its position at time t is $x(t)$. The ball feels the acceleration $x''(t) = -f'(x)$. (The sign is chosen as given because for a positive slope $f'(x) > 0$ the ball feels a negative acceleration and for a negative slope $f'(x) < 0$, the ball is accelerated.) If the motion of the ball is described in coordinates $(x(t), y(t))$, where $y(t) = x'(t)$ is the velocity of the ball, the corresponding vector field is $\vec{F}(x,y) = \langle y, -f'(x) \rangle$. Because $\vec{r}\,'(t) = \langle x'(t), y'(t) \rangle = \langle y(t), -f'(x(t)) \rangle$, the flow lines $\vec{r}(t) = \langle x(t), y(t) \rangle$ describe the position and velocity of the ball at time t. Draw the vector field $F(x,y)$, draw a few typical flow lines in the plane and match these curves with the corresponding motion of the ball.

Hint. It helps to look at the places, where the vector field is zero. These are called "equilibrium points" and correspond to situations, where the ball does not move.



Solution:
The point $(1/\sqrt{2}, 0)$ in the phase space is a point, where the vector field is zero. Around this, points, there are flow lines are circular. They correspond to the ball oscillating in the right well. A similar picture appears around $(-1/\sqrt{2}, 0)$, which corresponds to the case, when the ball bounces around in the left well, not having enough energy to leave. There are big circular flow lines, which correspond to the situation when the ball bounces from the very left to the right and back, having more energy than necessary to take the "bump" in the middle. There is one special flow line which looks like a figure 8. It corresponds to the situation, when the ball has just enough energy to reach the top of the hill in the middle. Where ever you release a ball with that energy, it will end up in the point $(0,0)$. To the orientation of the arrows. On the upper half plane, the velocity y is positive. This means that the ball moves to the right in that case. Therefore, the arrows point to the right on the upper half plane. On the lower half plane, where the velocity is negative, the ball moves to the left and the arrows point to the left.



5) (vector field) The vector field

$$\vec{F}(x,y,z) = \langle 5x^4 y + z^4 + y * \cos(x*y), x^5 + x * \cos(x*y), 4xz^3 \rangle$$

is a gradient field. Find the potential function f.

Section 5.4: Fundamental theorem of line integrals

1) (line integrals) Let C be the space curve $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ for $t \in [0,1]$ and let $\vec{F}(x,y,z) = \langle y, x, 5 \rangle$. Calculate the line integral $\int_C \vec{F} \cdot \vec{dr}$.

Solution:
$\int_C \vec{F} \cdot \vec{dr} = \int_0^1 \langle \sin(t), \cos(t), 5 \rangle \cdot \langle -\sin(t), \cos(t), 1 \rangle \; dt = \int_0^1 (\cos(2t) + 5) \; dt = \sin(2)/2 + 5$.

2) (line integrals) Find the work done by the force field $F(x,y) = (x\sin(y), y)$ on a particle that moves along the parabola $y = x^2$ from $(-1,1)$ to $(2,4)$.

Solution:
We parametrize the curve C by $\vec{r}(t) = \langle t, t^2 \rangle$ so that $\vec{r}\,'(t) = \langle 1, 2t \rangle$.

$$W = \int_C \vec{F} \cdot \vec{dr} = \int_{-1}^2 \langle t\sin(t^2), t^2 \rangle \cdot \langle 1, 2t \rangle \; dt = \int_{-1}^2 (t\sin(t^2) + 2t^3) \; dt = \cos(1)/2 - \cos(4)/2 + 15/2 \; .$$

3) (line integrals) Let $\vec{F}$ be the vector field $\vec{F}(x,y) = \langle -y, x \rangle/2$. Compute the line integral of F along an ellipse $\vec{r}(t) = \langle \cos(t), b\sin(t) \rangle$ with width $2a$ and height $2b$. The result should depend on a and b.

Solution:
The velocity is $\vec{r}'(t) = \langle -a\sin(t), b\cos(t) \rangle$ and $\vec{F}(\vec{r}(t)) = \langle -b\sin(t), a\cos(t) \rangle/2$ so that $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = ab/2$. If we integrate this from 0 to 2π we get the result πab.

4) (line integrals) After this summer school, you relax in a Jacuzzi and move along curve C which is given by part of the curve $x^{10} + y^{10} = 1$ in the first quadrant, oriented counter clockwise. The hot water in the tub has the velocity $\vec{F}(x,y) = \langle x, y^4 \rangle$. Calculate the line integral $\int_C \vec{F} \cdot \vec{dr}$, the energy you gain from the fluid force.

Solution:
Actually, this is a conservative field because $F(x,y) = \nabla f(x,y)$ with $f(x,y) = x^2/2 + y^5/5$ so that the line integral is zero.
However, we can also compute the line integral: $\vec{r}(t) = (4\cos(t), 4\sin(t))$ parametrizes the circle. $\vec{r}'(t) = (-4\sin(t), 4\cos(t))$ and $F(r(t)) = (4\cos(t), 256\sin^4(t))$ so that $\int_0^{2\pi} F(r(t)) \cdot r'(t) \; dt = \int_0^{2\pi} (-16\sin(t)\cos(t) + 1024\sin^4(t)\cos(t)) \; dt = (8\sin^2(t) + 1024\sin^5(t)/5t)|_0^{2\pi} = 0$.

5) (line integrals) Find a closed curve $C : \vec{r}(t)$ for which the vector field

$$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle = \langle xy, x^2 \rangle$$

satisfies $\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \; dt \neq 0$.

Solution:
We knew already that, because $Q_x = 2x$ and $P_y = x$, the field can not be a gradient field $\langle P, Q \rangle = \langle f_x, f_y \rangle$. Any curve not symmetric with respect to the y axes should work, for example a circle centered at $(1,1)$.