

Homework for Chapter 4. Extrema and Double integrals

Section 4.1: Extrema

- 1) (extrema) Find all the extrema of the function $f(x, y) = x^3 - 3x + 2y^2 - y^4$ and determine whether they are maxima, minima or saddle points.

Solution:

The condition $\nabla f(x, y) = (f_x, f_y) = (0, 0)$ are the equations

$$3x^2 - 3 = 0 \tag{1}$$

$$4y - 4y^3 = 0 \tag{2}$$

Point	D	f_{xx}	nature
(-1,-1)	48	-6	local maximum
(-1,0)	-24	-6	saddle
(-1,1)	48	-6	local maximum
(1,-1)	-48	6	saddle point
(1,0)	24	6	local minimum
(1,1)	-48	6	saddle point

have the solution $x = \pm 1, y = 0$ or $y = \pm 1$.

- 2) (extrema) Where on the parametrized surface $\vec{r}(u, v) = \langle u^2, v^3, uv \rangle$ is the temperature $T(x, y, z) = 12x + y - 12z$ minimal? To find the extrema, determine where the function $f(u, v) = T(\vec{r}(u, v))$ has local maxima, local minima or saddle points.

Hint. After you have found $f(u, v)$, you can rename the variables $f(x, y)$ if you like and start fresh.

Solution:

The function is $12x^2 + y^3 - 12xy$. The condition $\nabla f(x, y) = \langle f_x, f_y \rangle = (0, 0)$ are the equations

$$24x - 12y = 0 \tag{3}$$

$$-12x + 3y^2 = 0 \tag{4}$$

which has the solution $(0, 0), (1, 2)$. The data are

point (a,b)	D	f_{xx}	nature	$f(a, b)$
(0,0)	-144	24	saddle	0
(1,2)	144	24	minimum	-4

- 3) (extrema) Find and classify all the extrema of the function $f(x, y) = e^{-x^2-y^2}(x^2 + 2y^2)$.

Solution:

The equations $\nabla f = \langle (2^2 + 2y^2 - 1)xe^{-x^2-y^2}, (x^2 + 2y^2 - 2)ye^{-x^2-y^2} \rangle = \langle 0, 0 \rangle$ have the solutions $x = 0, y = 0, x = 0, y = \pm 1, y = 0, x = \pm 1$. By evaluating the discriminant D and f_{xx} at each point we find their nature:

Point	D	f_{xx}	nature
(0,0)	8	2	local minimum
(0,1)	$16/e^2$	$-2/e$	local maximum
(0,-1)	$16/e^2$	$-2/e$	local maximum
(1,0)	$-8/e^2$	$-4/e$	saddle point
(-1,0)	$-8/e^2$	$-4/e$	saddle point

- 4) (global extrema) Find all extrema of the function $f(x, y) = x^3 + y^3 - 3x - 12y + 20$ on the plane and characterize them. Do you find a global maximum or global minimum among them?

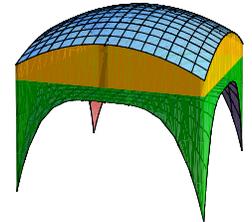
Solution:

The critical points satisfy $\nabla f(x, y) = \langle 0, 0 \rangle$ or $\langle 3x^2 - 3, 3y^2 - 12 \rangle = \langle 0, 0 \rangle$. There are 4 critical points $(x, y) = (\pm 1, \pm 2)$. The discriminant is $D = f_{xx}f_{yy} - f_{xy}^2 = 36xy$ and $f_{xx} = 6x$.

point	D	f_{xx}	classification	value
(-1,-2)	72	-6	maximum	38
(-1, 2)	-72	-6	saddle	6
(1, -2)	-72	6	saddle	34
(1, 2)	72	6	minimum	2

Note that there are no global (= absolute) maxima nor global minima because the function takes arbitrarily large and small values. For $y = 0$ the function is $g(x) = f(x, 0) = x^3 - 3x + 20$ which satisfies $\lim_{x \rightarrow \pm\infty} g(x) = \pm\infty$.

- 5) (extrema) The thickness of the region enclosed by the two graphs $f_1(x, y) = 10 - 2x^2 - 2y^2$ and $f_2(x, y) = -x^4 - y^4 - 2$ is denoted by $f(x, y) = f_1(x, y) - f_2(x, y)$. Classify all critical points of f and find the global minimal thickness.



Solution:

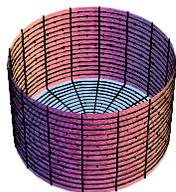
The function to extremize is $f(x, y) = 12 + x^4 + y^4 - 2x^2 - 2y^2$. Its gradient is $\nabla f(x, y) = \langle 4x^3 - 4x, 4y^3 - 4y \rangle$. This gradient is equal to $\langle 0, 0 \rangle$ if $x \in \{0, 1, -1\}$ and $y \in \{0, 1, -1\}$. There are 9 critical points. Now we proceed and use the second derivative test. We compute the discriminant $D = f_{xx}f_{yy} - f_{xy}^2 = (12x^2 - 4)(12y^2 - 4)$ and $f_{xx} = 12x^2 - 4$. D is negative if exactly one of the x, y is zero. Otherwise, it is positive. f_{xx} is negative

x	y	D	f_{xx}	Type	$f(x,y)=$
-1	-1	64	8	minimum	10
-1	0	-32	8	saddle	11
-1	1	64	8	minimum	10
0	-1	-32	-4	saddle	11
0	0	16	-4	maximum	12
0	1	-32	-4	saddle	11
1	-1	64	8	minimum	10
1	0	-32	8	saddle	11
1	1	64	8	minimum	10

if $x = 0$. The minimal value 10 occurs at 4 places. These are $\boxed{(-1, -1), (-1, 1), (1, -1), (1, 1)}$. These are local minima. But they are also global minima because $f(x, y) = (x^2 - 1)^2 + (y^2 + 1)^2 + 10$ is always positive and goes to infinity for $(x, y) \rightarrow \infty$.

Section 4.2: Lagrange

- 1) (Lagrange) Find the cylindrical basket which is open on the top has the largest volume for fixed area π . If x is the radius and y is the height, we have to extremize $f(x, y) = \pi x^2 y$ under the constraint $g(x, y) = 2\pi xy + \pi x^2 = \pi$. Use the method of Lagrange multipliers.

**Solution:**

The Lagrange equations are

$$\begin{aligned} 2xy\pi &= (2x\pi + 2y\pi)\lambda \\ \pi x^2 &= 2\pi x\lambda \\ \pi x^2 + 2\pi xy &= \pi \end{aligned}$$

Since $x = 0$ is not possible (it would violate the constraint), we can divide the second equations by x and divide the first by the second equation. This gives $\boxed{x = y = 1/\sqrt{3}}$.

The maximum value is $\boxed{\pi\sqrt{3}/9}$.

- 2) (global extrema with Lagrange) Find the extrema of the same function $f(x, y) = e^{-x^2-y^2}(x^2+2y^2)$ as in problem 3.1.3 but now on the entire disc $\{x^2 + y^2 \leq 4\}$ of radius 2. Besides the already found extrema inside the disk, you have to find extrema on the boundary.

Solution:

The Lagrange equations are

$$\begin{aligned} (2x - 2x^3 - 4xy^2)e^{-x^2-y^2} &= \lambda 2x \\ (4y - 4y^3 - 2x^2y)e^{-x^2-y^2} &= \lambda 2y \\ x^2 + y^2 &= 4. \end{aligned}$$

Case 1: If $x = 0$, then the first equation is ok and we get from the third equation $y = \pm 2$.

Case 2: If $y = 0$, then the second equation is ok and we get from the third equation $x = \pm 2$.

Case 3: If $x = 0$ and $y = 0$, then the first two equations are ok, but clashes with the third. Forget this case.

Case 4: If both x and y are not zero we can divide the first equation by $2x$ and the second by $2y$. We also replace $-x^2 - y^2$ by -4

$$\begin{aligned} (1 - x^2 - 2y^2)e^{-4} &= \lambda \\ (2 - 2y^2 - x^2)e^{-4} &= \lambda \\ x^2 + y^2 &= 4. \end{aligned}$$

But setting the first two equations equal leads to a contradiction. Also this case 4) has no solutions. We end up with the four solutions $(2, 0), (-2, 0), (0, 2), (0, -2)$. The minimal values are $f(\pm 2, 0) = 4e^{-4}$, the maximal values are $f(0, \pm 2) = 8e^{-4}$. We can now make a list of all the candidates for extrema

1. Extrema inside

2. Extrema on the boundary

point	$f =$
(0,0)	0
(1,0)	$1/e$
(-1,0)	$1/e$
(0,1)	$2/e$
(0,-1)	$2/e$

point	$f =$
(2,0)	$4/e^4$
(-2,0)	$4/e^4$
(0,2)	$8/e^4$
(0,-2)	$8/e^4$

We see that the origin is the minimum and the points $(0, \pm 1)$ are both the maxima.

- 3) Find the points (x, y, z) on the surface $g(x, y, z) = xy^2 - z^3 - 2 = 0$ that are closest to the origin $(0, 0, 0)$.

Solution:

Instead of extremizing the distance $\sqrt{x^2 + y^2 + z^2}$, we extremize the function $f(x, y) = x^2 + y^2 + z^2$. We have the Lagrange equations

$$\begin{aligned} 2x &= \lambda y^2 \\ 2y &= \lambda 2xy \\ 2z &= -\lambda 3z^2 \\ xy^2 &= z^3 + 2 \end{aligned}$$

1. **Case:** $z = 0$. We can then not have $y = 0$ nor $x = 0$ and end up with

$$\begin{aligned} 2x &= \lambda y^2 \\ 2 &= \lambda 2x \\ xy^2 &= 2 \end{aligned}$$

which gives $2 = \lambda^2 y^2$, $x^2 = \lambda$ and so that $x = 1$, $y = \pm\sqrt{2}$.

2. **Case:** $x = 0$. Gives $y = 0$ and $z = -2^{1/3}$.

3. **Case:** $y = 0$. Gives $x = 0$ and $z = -2^{1/3}$.

4. **Case:** all x, y, z are nonzero. Then

$$\begin{aligned} 2x &= \lambda y^2 \\ 1 &= \lambda x \\ 2 &= -\lambda 3z \\ xy^2 &= z^3 + 2 \end{aligned}$$

Eliminating $\lambda = 1/x$ gives

$$\begin{aligned} 2x^2 &= y^2 \\ 2 &= -3z/x \\ xy^2 &= z^3 + 2 \end{aligned}$$

Solving for $y^2 = 2x^2$ and $z = -2x/3$ from the first two equations and plugging this into the third gives $2x^3 = -8x^3/27 + 2$ gives $x = 3/31^{1/3}$ and so $y = \pm\sqrt{2}(3/31^{1/3})$.

The distance from the point $(0, 0, -2^{1/3})$ to the origin is $2^{1/3} \sim 1.25$ the distance from the points $(3/31^{1/3}(1, \pm\sqrt{2}, -2/3))$ to the origin is $31^{1/6} \sim 1.7723$. The first one is the minimum.

- 4) Let a, b, c be non-negative constants and let F be the function $F(x, y, z) = -x \log(x) - y \log(y) - z \log(z) - ax - by - cz$. Find the maxima and minima of F on $x > 0, y > 0, z > 0$ under the constraint $x + y + z = 1$.

Remark: This problem appears in thermodynamics and is relevant in biology or chemistry. If x, y, z are the probabilities that a system is in state X, Y, Z and a, b, c are the energies for these states. Then $-x \log(x) - y \log(y) - z \log(z)$ is called the **entropy** of the system and $E = ax + by + cz$ is the **energy**. The number $F(x, y, z)$ is called the **free energy**. If energy is fixed, nature tries to maximize entropy. Otherwise it tries to **minimize the free energy** $F = S - E$. If we extremize F under the constraint of having total probability $G(x, y, z) = x + y + z = 1$, we obtain the so called **Gibbs distribution**.

Solution:

The Lagrange equations are

$$\begin{aligned} -\log(x) - 1 - a &= \lambda \\ -\log(y) - 1 - b &= \lambda \\ -\log(z) - 1 - c &= \lambda \\ x + y + z &= 1 \end{aligned}$$

From the first three equations, we get

$$\begin{aligned} x = e^{-(1+a+\lambda)} &= e^{-1-\lambda} e^{-a} \\ y = e^{-(1+b+\lambda)} &= e^{-1-\lambda} e^{-b} \\ z = e^{-(1+c+\lambda)} &= e^{-1-\lambda} e^{-c} \end{aligned}$$

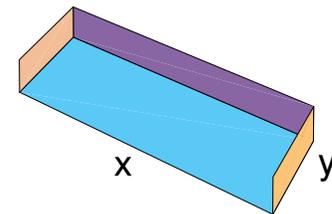
Plugging this into the fourth equation gives $e^{-1-\lambda}(e^{-a} + e^{-b} + e^{-c}) = 1$ so that $e^{-1-\lambda} = (e^{-a} + e^{-b} + e^{-c})^{-1}$ and

$$\begin{aligned} x &= e^{-a} / (e^{-a} + e^{-b} + e^{-c}) \\ y &= e^{-b} / (e^{-a} + e^{-b} + e^{-c}) \\ z &= e^{-c} / (e^{-a} + e^{-b} + e^{-c}) \end{aligned}$$

- 5) Minimize the material cost of an office tray

$$f(x, y) = xy + x + 2y$$

of length x , width y and height 1 under the constraint that the volume $g(x, y) = xy$ is constant and equal to 4.



Solution:

The Lagrange equations for the function given are

$$\begin{aligned}y + 1 &= \lambda y \\x + 2 &= \lambda x \\xy &= 4\end{aligned}$$

Because $x = 0$ and $y = 0$ are both not compatible with the third equation, we can divide by x and y . Dividing the first by the second equation, gives

$$(y + 1)/(x + 2) = y/x$$

which leads to the minimum $(x, y) = (2\sqrt{2}, \sqrt{2})$.

Section 4.3: Double integrals

- 1) (double integral) Calculate the iterated integral $\int_1^4 \int_0^2 (2x - \sqrt{y}) \, dx \, dy$. Interpret the result in terms of volumes.

Solution:

Start with the inner integral $\int_0^2 (2x - \sqrt{y}) \, dx = 4 - 2\sqrt{y}$. Integrating this from 1 to 4 gives $\frac{8}{3}$. The graph of $2x = \sqrt{y}$ passes through the rectangle so that the function is positive somewhere in the rectangle and negative elsewhere. The integral is not a volume but the difference between two positive volumes.

- 2) (double integral) Find the area of the region

$$R = \{(x, y) \mid 0 \leq x \leq 2\pi, \sin(x) - 1 \leq y \leq \cos(x) + 2\}$$

and use it to compute the average value of $f(x, y) = y$ over that region.

Remark. You will use here the integral $\int_0^{2\pi} \sin^2(x) \, dx$ treated in class.

Solution:

In this problem, it helps to see that $\int_0^{2\pi} \cos(x) \, dx = 0$ and $\int_0^{2\pi} \cos^2(x) \, dx = \pi$ and the same for \sin .

The area is $A = \int_0^{2\pi} \int_{\sin(x)-1}^{\cos(x)+2} 1 \, dx \, dy = \int_0^{2\pi} (\cos(x) + 2) - (\sin(x) - 1) \, dx = 6\pi$. The average value is $\int_0^{2\pi} \int_{\sin(x)-1}^{\cos(x)+2} y \, dx \, dy / A = \int_0^{2\pi} (\cos(x) + 2)^2 - (\sin(x) - 1)^2 \, dx / A = (4 - 1)\pi / (2\pi) = 3\pi / (6\pi) = \frac{1}{2}$.

- 3) (volume) Find the volume of the solid lying under the paraboloid $z = x^2 + y^2$ and above the rectangle $R = [-2, 2] \times [-3, 3] = \{(x, y) \mid -2 \leq x \leq 2, -3 \leq y \leq 3\}$.

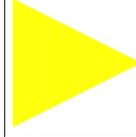
Solution:

We have to compute the double integral of $f(x, y) = x^2 + y^2$ over R . The inner integral is $\int_{-3}^3 (x^2 + y^2) \, dx = 18 + 6y^2$ so that $\int_{-2}^2 \int_{-3}^3 (x^2 + y^2) \, dy \, dx = \int_{-2}^2 (18 + 6y^2) \, dy = 104$.

- 4) (switching order of integration) Calculate the iterated integral $\int_0^1 \int_x^{2-x} (x^2 - y) \, dy \, dx$. Sketch the corresponding type I region. Write this integral as integral over a type II region and compute the integral again.

Solution:

$\int_0^1 \int_x^{2-x} (x^2 - y) \, dy \, dx = -5/6$. The region is a triangle bound by the lines $y = x$, the line $y = 2 - x$ and the y axis. The inner integral is $-2 + 2x + 2x^2 - 2x^3$.



As a type II region, the region has to be split $\int_0^1 \int_0^y (x^2 - y) \, dx \, dy + \int_1^2 \int_0^{2-x} (x^2 - y) \, dx \, dy = -1/4 - 7/12 = -5/6$.

- 5) (double integral) Evaluate the double integral

$$\int_0^2 \int_{x^2}^4 \frac{x}{e^{y^2}} \, dy \, dx.$$

Solution:

Change the order of integration:

$$\int_0^4 \int_0^{\sqrt{y}} \frac{x}{e^{y^2}} \, dx \, dy = -\frac{1}{4} e^{-y^2} \Big|_0^4 = \frac{1 - e^{-16}}{4}.$$

It can also be written as $\frac{(e^{16} - 1)}{(4e^{16})}$.

Section 4.4: Polar integration

- 1) (polar integrals) Integrate $f(x, y) = x^2$ over the unit disc $\{x^2 + y^2 \leq 1\}$ in two ways, first using Cartesian coordinates, then using polar coordinates.

Solution:

The integral in Cartesian coordinates goes less smooth because we have to compute a 1D integral with partial integration: $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2 \, dy \, dx = \int_{-1}^1 2x^2 \sqrt{1-x^2} \, dx = \pi/4$.

The integral in polar coordinates is easier to get: using the substitution $x = \cos(u)$ we obtain

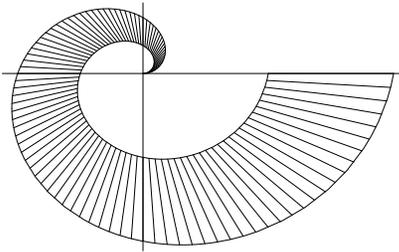
$$\int_0^{2\pi} \int_0^1 r^2 \cos(\theta)^2 r \, dr \, d\theta = (1/4)\pi.$$

- 2) (polar integrals) Find $\int \int_R (x^2 + y^2)^{10} \, dA$, where R is the part of the unit disc $\{x^2 + y^2 \leq 1\}$ for which $y > x$.

Solution:

The solution is easier in polar coordinates: $\int_0^1 \int_{\pi/4}^{5\pi/4} r^{21} \, d\theta \, dr = \pi/22$. This integral would give quite a bit of work to solve in Cartesian coordinates.

- 3) (polar integrals) What is the area of the region which is bounded by three curves, first by the polar curve $r(\theta) = \theta$ with $\theta \in [0, 2\pi]$, second by the polar curve $r(\theta) = 2\theta$ with $\theta \in [0, 2\pi]$ and third by the positive x -axis.



Solution:

$$\int_0^{2\pi} \int_{\theta}^{2\theta} r \, dr d\theta = \int_0^{2\pi} (2\theta)^2/2 - (\theta)^2/2 \, d\theta = (3/2) \int_0^{2\pi} \theta^2 \, d\theta = (3/2)(2\pi)^3/3 = 4\pi^3.$$

- 4) (polar integrals) Find the average value of $f(x, y) = x^2 + y^2$ on the annulus $1 \leq |(x, y)| \leq 2$.

Solution:

The integral of f over the annulus is $2\pi \int_1^2 r^3 \, dr = 2\pi(16 - 1)/4$. The area is $2\pi \int_1^2 r \, dr = 2\pi(4 - 1)/2$. The average is $(15/4)/(3/2) = 5/2$.

- 5) (surface area) Find the surface area of the paraboloid $x = y^2 + z^2$ that lies inside the cylinder $y^2 + z^2 = 9$.

Solution:

We use polar coordinates in the yz -plane. The paraboloid is parametrized by $(u, v) \mapsto (v^2, v \cos(u), v \sin(u))$ and the surface integral $\int_0^3 \int_0^{2\pi} |\vec{r}_u \times \vec{r}_v| \, dudv$ is equal to $\int_0^3 \int_0^{2\pi} v\sqrt{1+4v^2} \, dudv = 2\pi \int_0^3 v\sqrt{1+4v^2} \, dv = \pi(37^{3/2} - 1)/6$.