

Homework for Chapter 3. Linearization and Gradient

Section 3.1: Partial derivatives and partial differential equations

- 1) (partial differential equations) Verify that $u(t, x) = \sin(\sin(t + x))$ is a solution of the transport equation $u_t(t, x) = u_x(t, x)$.

Solution:

This is a direct computation. Just differentiate $u(t, x)$ with respect to t and x which both give $\cos(\sin(t + x)) \cos(t + x)$.

- 2) (partial differential equations) Verify that $f(x, y) = 3y^2 + x^3$ satisfies the **Euler-Tricomi** partial differential equation $u_{xx} = xu_{yy}$. This partial differential equation is useful in describing transonic flow. Can you find an other solution which is not a multiple of the solution given in this problem?



Solution:

Just take derivatives. Yes $y^3 + x^3y$ or $6xy^2 + x^4$ are other solutions.

- 3) (partial differential equations) Verify that $f(x, t) = e^{-rt} \sin(x + ct)$ satisfies the PDE $f_t(x, t) = cf_x(x, t) - rf(x, t)$ called the **advection equation**.

Solution:

Differentiate $f_x(x, y) = e^{-rt} \cos(x + ct)$ and $f_t(x, y) = -re^{-rt} \sin(x + ct) + ce^{-rt} \cos(x + ct)$ and compare.

- 4) (partial derivatives) The partial derivatives of the function $f(x, y) = (xy)^{1/3}$ exists at $(0, 0)$ but the directional derivatives in all other directions don't exist at the point $(0, 0)$. What is going on?

Solution:

The function is not smooth at the origin. The partial derivatives exist at $(0, 0)$ by "accident" because the function is 0 on the x axes is constant 0. Since the gradient is zero at $(0, 0)$ we could conclude that $D_v f = \nabla f \cdot v$ is zero for all directions. But the directional derivatives do not exist. For example, for $v = \langle 1, 1 \rangle / \sqrt{2}$ the limit $\lim_{h \rightarrow 0} (f(h, h) - f(0, 0)) / h$ does not exist.

- 5) (partial differential equations)

Let $g(x, y)$ be the distance from a point (x, y) to the curve $x^2 + 2y^2 + y^4/10 = 1$. Show that g is a solution of the partial differential equation

$$f_x^2 + f_y^2 = 1$$

outside the curve. No computations are needed. The shape of the curve is pretty much irrelevant. What does the PDE say about the gradient ∇f ?

Remark: This problem only needs thought. Use it as a "pillow problem" that is think about it before going to sleep. By the way, the PDE is called **eiconal equation**. It describes wave fronts in optics.

Solution:

If you move perpendicular to a level curve, then your function changes according to the length of the gradient. If the gradient is large, then the function changes a lot, if the gradient is small, then the function changes little. The given partial differential equation tells that that the gradient of f has length 1. This means that if go along the steepest decent directions all the time, then you descend by the exactly same amount with which you go away from the level curve. So, the level curve of the height function $f(x, y)$ is equal to the level curve of the distance function $g(x, y)$.

While the solution to the problem is difficult to explain, it essentially just asks you to see what are the consequences of having a gradient of length 1 everywhere. If you have a mountain of height $f(x, y)$ for which the gradient is 1 everywhere, it has the property that if you decent always perpendicular to the level curve (the fastest decent), then the height decreases in the same way as the distance you go away (the slope is 1 in that direction, the directional derivative is 1 in the gradient direction).

If you are still confused, look at the one dimensional problem: the function $f(x)$ giving the distance to an interval $[a, b]$ satisfies the differential equation $|f'(x)|^2 = 1$. The reasoning is almost the same and if you can answer this, you will understand the 2 dimensional problem better too.

Section 3.2: Linear approximation and tangents

- 1) (tangent planes) If $2x + 3y + 2z = 9$ is the tangent plane to the graph of $z = f(x, y)$ at the point $(1, 1, 2)$. Estimate $f(1.01, 0.98)$

Solution:

The approximation must lie on the tangent plane. Plug in $x = 1.01, y = 0.98$. We get $z = (9 - 2 * 1.01 - 3 * 0.98)/2 = 2.02$.

- 2) (estimation) Estimate $1000^{1/5}$ using linear approximation.

Solution:

The function $f(x) = x^{1/5}$ has the deriv the root is close to 2, the value of $f(x) : 2 - 24/(5 \cdot 2^4)$.

- 3) (estimation) Find $f(0.01, 0.999)$ for $f(x, y) = \cos(\pi xy)y + \sin(x + \pi y)$

Solution:

Approximate $f(x, y)$ by $L(x, y) = f(0, 1) + \nabla f(0, 1) \cdot (x - 0, y - 1)$ and then compute $L(0+0.01, 1-0.0001)$. We have $\nabla f(x, y) = (-\sin(\pi xy)\pi y^2 + \cos(x + \pi y), -\sin(\pi xy)\pi xy + \cos(\pi xy) + \pi \cos(x + \pi y))$ so that $\nabla f(0, 1) = (-1, 1 - \pi)$ Finally $L(x, y) = 1 + f_x(0, 1)x + f_y(0, 1)(y - 1) = 1 - x - (1 - \pi)y$ and therefore $L(0.01, 1 - 0.0001) = 0.989 + \pi \cdot 0.001$.

- 4) (linear approximation) Find the linear approximation $L(x, y)$ of the function $f(x, y) = \sqrt{10 - x^2 - 5y^2}$ at $(2, 1)$ and use it to estimate $f(1.95, 1.04)$.

Solution:

a) $f(2, 1) = 1, \nabla f(x, y) = (-2x, -10y)/(2f(x, y)) = (-x, -5y) = (-2, -5)$. $L(x, y) = f(2, 1) + \nabla f(2, 1)(x - 2, y) = 1 + (-2, -5)(x - 2, y) = 1 - x + (1 - \pi)y$ so that $L(1.95, 1.04) = f(2, 1) + \nabla f(2, 1)(-0.05, 0.04) = 1 + (-2, -5)(-0.05, 0.04) = 1 + 0.10 - 0.2 = 0.90$ which can be compared with the actual value $f(1.95, 1.04) = 0.88888$. Note that the linear approximation could be computed without taking square roots.

b) $(-2, -5)(-3, 4)/5 = -14/5$.

- 5) (tangent lines) Sketch a contour map of the function $f(x, y) = x^2 + 9y^2$, find the **gradient vector** $\nabla f = \langle f_x, f_y \rangle$ of f at the point $(1, 1)$ and draw it together with the tangent line $ax + by = d$ to the curve at $(1, 1)$.

Solution:

a) The level curves $x^2 + 9y^2 = c$ are ellipses.
 b) $\nabla f(x, y) = (2x, 18y)$. At the point $(1, 1)$, it is the vector $(2, 18)$.
 c) The line has the form $2x + 18y = d$, where d is obtained by plugging in the coordinates of the point: $d = 20$. The equation is $x + 9y = 10$.
 d) The linear approximation is $L(x, y) = f(1, 1) + (2, 18)(x - 1, y - 1) = 10 + 2 \cdot 0.001 - 8 \cdot 0.001 = 10 - 0.06 = 9.984$.

Section 3.3: Chain rule and implicit differentiation

- 1) (chain rule) You know that $d/dt f(\vec{r}(t)) = 2$ if $\vec{r}(t) = \langle t, t \rangle$ and $d/dt f(\vec{r}(t)) = 3$ if $\vec{r}(t) = \langle t, -t \rangle$. Find the gradient of f at $(0, 0)$.

Solution:

Let $\nabla f = \langle a, b \rangle$ denote the gradient. Then the assumptions give $a + b = 2$ and $a - b = 3$. Therefore $a = 5/2, b = -1/2$.

- 2) (chain rule) The pressure in the space at the position (x, y, z) is $p(x, y, z) = x^2 + y^2 - z^3$ and the trajectory of an observer is the curve $\vec{r}(t) = \langle t, t, 1/t \rangle$. Using the chain rule, compute the rate of change of the pressure the observer measures at time $t = 2$.

Solution:

a) The multivariable chain rule is

$$\frac{d}{dt} p(\vec{r}(t)) = \nabla p(\vec{r}(t)) \cdot \vec{r}'(t).$$

b) $\nabla p(x, y, z) = \langle 2x, 2y, -3z^2 \rangle, \vec{r}'(t) = \langle 1, 0, -1/t^2 \rangle$. We have $\vec{r}(2) = \langle 1, 1, 1/2 \rangle$ and $\vec{r}'(2) = \langle 1, 1, -1/4 \rangle$. By the chain rule in a), we have

$$\nabla p(1, 1, 1/2) \cdot \vec{r}'(2) = 4 - 3/8.$$

- 3) (chain rule) Mechanical systems are determined by the energy function $H(x, y)$, a function of two variables. The first, x is the position and the second y is the momentum. The equations of motion for the curve $\vec{r}(t) = \langle x(t), y(t) \rangle$ are

$$\begin{aligned} x'(t) &= H_y(x, y) \\ y'(t) &= -H_x(x, y) \end{aligned}$$

They are called called **Hamilton equations**. a) Using the chain rule, verify that in full generality, the energy of a Hamiltonian system is preserved: for every path $\vec{r}(t) = \langle x(t), y(t) \rangle$ solving the system, we have $H(x(t), y(t)) = const$.

b) Check this in the particular case of the **pendulum**, where $H(x, y) = y^2/2 - \sin(x)$.

- 4) (implicit differentiation single variable) Derive using implicit differentiation the derivative $d/dx \arctanh(x)$, where $\tanh(x) = \sinh(x)/\cosh(x)$.

Reminder: the definitions of the **hyperbolic sine** and **hyperbolic cosine** are $\sinh(x) = (e^x - e^{-x})/2$ and $\cosh(x) = (e^x + e^{-x})/2$. Note that $\sinh' = \cosh$ and $\cosh' = \sinh$ and $\cosh^2(x) - \sinh^2(x) = 1$.

Solution:

From $\cosh^2(x) - \sinh^2(x) = 1$ follows $1 - \tanh^2(x) = 1/\cosh^2(x)$. $d/dx \tanh(\arctanh(x)) = 1/\cosh^2(\arctanh(x)) \tanh'(x) = x$ Now $1/\cosh^2(x) = 1/(1 - \tanh^2(x))$ so that $\tanh'(x) = 1/(1 - x^2)$.

- 5) (implicit differentiation) The equation $f(x, y, z) = e^{xyz} + z = 1 + e$ implicitly defines z as a function $z = g(x, y)$ of x and y . Find formulas (in terms of x, y and z) for $g_x(x, y)$ and $g_y(x, y)$. and Estimate $g(1.01, 0.99)$ using linear approximation.

Solution:

By the chain rule $g_x = -f_x/f_z = -yze^{xyz}/(1 + xye^{xyz})$ and $g_y = -f_y/f_z = -xze^{xyz}/(1 + xye^{xyz})$.

Note that $f(1, 1, 1) = 1 + e$ so that $(1, 1, 1)$ is on the surface and $g(1, 1) = 1$. From a) we know $g_x(1, 1) = -e/(1 + e)$ and $g_y(1, 1) = -e/(1 + e)$. By linearization, $g(1 + a, 1 + b) = g(1, 1) + g_x(1, 1)a + g_y(1, 1)b = 1 - (a + b)e/(1 + e)$. In our case, where $a = 0.01, b = -0.01$ we estimate $g(1.01, 0.99) = 1$.

Section 3.4: Gradient and directional derivative

- 1) (gradient vector) A surface $x^2 + y^2 - z = 1$ radiates light away. It can be parametrized as $\vec{r}(x, y) = \langle x, y, x^2 + y^2 - 1 \rangle$. Find the parametrization of the wave front which is distance 1 from the surface.

Solution:

The normal vector is $\nabla f(x, y) = \langle 2x, 2y, -1 \rangle$. Normalized, this is $\vec{n}(x, y) = \langle 2x, 2y, -1 \rangle / \sqrt{4x^2 + 4y^2 + 1}$. The wave front is $\vec{R} (= \vec{r}(x, y) + \vec{n}(x, y) = \langle x, y, x^2 + y^2 - 1 \rangle + \langle 2x, 2y, -1 \rangle / \sqrt{4x^2 + 4y^2 + 1}$.

- 2) (directional derivative) Find the directional derivative $D_{\vec{v}}f(2, 1) = \nabla f(2, 1) \cdot \vec{v}$ into the direction $\vec{v} = \langle -3, 4 \rangle / 5$ for the function $f(x, y) = x^5y + y^3 + x + y$.

Solution:

$\nabla f(x, y) = \langle 5x^4y + 1, x^5 + 3y^2 + 1 \rangle = \langle 81, 68 \rangle$. Therefore $D_{\vec{v}}f(2, 1) = 81(-3/5) + 68(4/5) = 29/5$.

- 3) (directional derivative) Assume $f(x, y) = 1 - x^2 + y^2$. Compute the directional derivative $D_{\vec{v}}f(x, y)$ at $(0, 0)$ where $\vec{v} = \langle \cos(t), \sin(t) \rangle$ is a unit vector. Now compute

$$D_{\vec{v}}D_{\vec{v}}f(x, y)$$

at $(0, 0)$, for any unit vector. For which directions is this **second directional derivative** positive?

Solution:

The directional derivative is $g(x, y) = \nabla f(x, y) \cdot \langle \cos(t), \sin(t) \rangle = \langle -2x \cos(t), 2y \sin(t) \rangle$. This is zero at $(0, 0)$. The second directional derivative is the directional derivative of g in the direction v . We have $\nabla g(x, y) = \langle -2 \cos(t), 2 \sin(t) \rangle$. Now, $D_{\vec{v}}g = -2 \cos^2(t) + 2 \sin^2(t) = -2 \cos(2t)$. This is negative on the intervals $(-\pi/4, \pi/4)$ and $(-3\pi/4, 5\pi/4)$ and positive on the intervals $(\pi/4, 3\pi/4)$ and $(5\pi/4, 7\pi/4)$.

- 4) (gradient) In the following two exercise we derive the so called **Kitchen-Rosenberg formula**

$$\kappa = \frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{(f_x^2 + f_y^2)^{3/2}}$$

for the curvature of a level curve $f(x, y) = c$ at a point (x_0, y_0) . It is used in computer vision.

Note first that the function

$$g(x, y) = \arctan(f_y/f_x)$$

is the angle of the gradient vector. Verify that the curvature of the level curve is the directional derivative of g in the direction $\vec{v} = \langle -f_y, f_x \rangle / \sqrt{f_x^2 + f_y^2}$.

Solution:

The curvature is the rate of change of the angle of the velocity vector when moving along the curve in unit speed. Since the gradient vector is always perpendicular, this is also the rate of change of the angle of the gradient vector.

- 5) (directional derivative) This is a continuation of the previous problem. Verify that

$$D_{\vec{v}}g(x, y, z)$$

is the expression κ above in the previous problem. You have now verified the Kitchen-Rosenberg formula.

Solution:

Simplify $\langle -f_y, f_x \rangle / \sqrt{f_x^2 + f_y^2} \cdot \langle \frac{\partial}{\partial x} / (1 + f_y^2/f_x^2), \frac{\partial}{\partial y} / (1 + f_y^2/f_x^2) \rangle$.