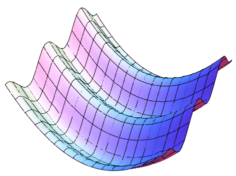
Oliver Knill, Harvard Summer school, Summer 2009

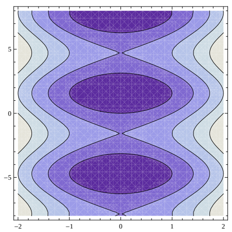
Homework for Chapter 2. Curves and Surfaces

Section 2.1: Functions, level surfaces, quadrics

1) (functions of two variables) Let $f(x, y) = y^2 - \sin(x)$. Find the equations for the three traces of the surface $g(x, y, z) = z - f(x, y) = 0$, the graph $z = f(x, y)$ of f. Sketch the surface.

Solution:
The contour map consists of circular regions for $c \in (-1, 0)$ or $y = \sqrt{c + \sin(x)}$ for $c > 0$.
For $c = 0$, we have the xy-trace, which consists of $y = \pm\sqrt{|\sin(x)|}$.
The yz-trace: $z = y^2$ is a "parabola".
The xz-trace: $z = -\sin(x)$ is the graph of the sin function.





2) (level surfaces) Consider the surface $z^2 - 4z + x^2 - 2x - y = 0$. Draw the three traces and What surface is it?

Solution:
Completion of the sphere gives $(z - 2)^2 + (x - 1)^2 - y = 5$. The surface is a paraboloid rotational symmetric parallel to the y axis. To see this, it is helpful to draw the generalized traces obtained by intersecting with $y = c$ which gives circles. Especially the intersection with the xy-plane is a circle. The other two traces are parabola.

3) (level surfaces) Surfaces satisfying the implicit equation $x^k + y^k = z^k$ with integer k are called **Fermat** surfaces.

a) Sketch the Fermat surface for $k = 2$ with traces.
b) Sketch the Fermat surface for $k = 4$ with traces.

Remark: You have found integer points (x, y, z) lying on the Fermat surface $x^2 + y^2 = z^2$ in a previous homework. It was Fermat, who conjectured first, that there are no nontrivial lattice points on the Fermat surfaces for $k > 2$. This claim is now a theorem.

Solution:
a) The surface $x^2 + y^2 = z^2$ is a cone.
b) The surface $x^4 + y^4 = z^4$ has on each height z a trace which has the shape when you deform a circle to a square.

4) a) Sketch the graph of the function $f(x, y) = \cos(x^2 + y^2)/(1 + x^2 + y^2)$.
b Sketch the graph of the function $g(x, y) = |x| - |y|$.
c) Sketch some contour curves $f(x, y) = c$ of f.
d) Sketch some contour curves $g(x, y) = c$ of g.

Solution:
a) The graph is rotational symmetric because the function depends only on $x^2 + y^2 = r^2$. It is a ripple in the pond.
b) In each quadrant, the graph is linear and a plane.
These four pieces come together above the coordinate axis.
c) These are ellipses.
d) The level curves are lines which break on the coordinate axes.

5) (level surfaces) Verify that the line $\vec{r}(t) = \langle 1, 3, 2\rangle + t\langle 1, 2, 1\rangle$ is contained in the surface $z^2 - x^2 - y = 0$.

Solution:
Plug in $x = 1 + t, y = 3 + 2t, z = 2 + t$ into the equation $(3 + 2t) = (2 + t)^2 - (1 + t)^2$.
The surface is actually a hyperbolic paraboloid.

Section 2.2: Parametric surfaces

1) (parametrized surfaces) Plot the surface with the parametrization $\vec{r}(u, v) = \langle v^2\cos(u), v^2\sin(u), v\rangle$, where $u \in [0, 2\pi]$ and $v \in \mathbf{R}$.

Solution:
It is a surface of revolution, very thin at the origin. The shape is a parabola but it is bent the other way round as in the paraboloid.

2) (parametrized surfaces) Find a parametrization for the plane which contains the points $P = (3, 7, 1)$,$Q = (1, 2, 1)$ and $R = (0, 3, 4)$.

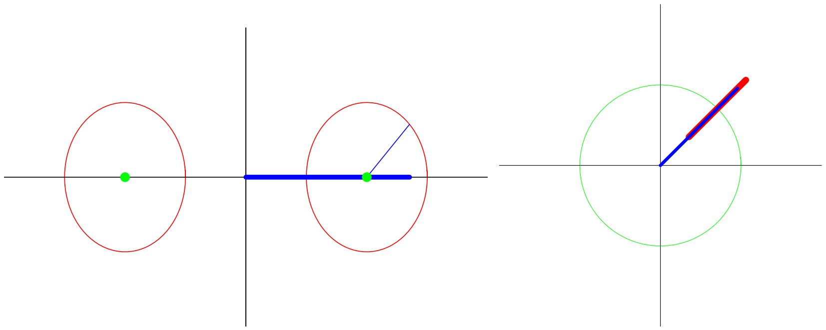
Solution:
Take $r(t) = P + s(Q - P) + t(R - P)$. $\vec{r}(s, t) = (3 - 2s - 3t, 7 - 5s - t, 1 + -3s + 3t)$.

3) (parametrized surfaces) Find two different parametrisations of the lower half of the ellipsoid $2x^2 + 4y^2 + z^2 = 1, z < 0$.
For the first parametrizations, assume that the surface is a graph $z = f(x, y)$. For the other, use angles ϕ, θ similarly for the sphere.

Solution:
Here are three possible parametrizations:
1) $\vec{r}(u, v) = (u, v, -\sqrt{1 - 2u^2 - 4v^2}$.
2) $\vec{r}(\theta, \phi) = (\sin(\phi)\cos(\theta)/\sqrt{2}, \sin(\phi)\sin(\theta)/2, \cos(\phi))$.

4) (parametrized surfaces) Find a parametrisation of the **torus** which is obtained as the set of points which have distance 1 from the circle $(2\cos(\theta), 2\sin(\theta), 0)$, where θ is the angle ocuring in cylindrical and spherical coordinates.

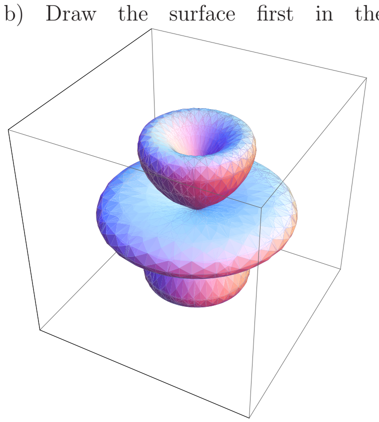
Hint: Keep $u = t$ as one of the parameters and let r the distance of a point on the torus to the z-axis. This distance is $r = 2 + \cos(\phi)$ if ϕ is the angle you see on Figure 1. You can read off from the same picture also $z = \sin(\phi)$. To finish the parametrization problem, you have to translate back from cylindrical coordinates $(r, \theta, z) = (2+\cos(\phi), \theta, \sin(\phi))$ to Cartesian coordinates (x, y, z). Write down your result in the form $\vec{r}(\theta, \phi) = (x(\theta, \phi), y(\theta, \phi), z(\theta, \phi))$.



Solution:
$\vec{r}(\theta, \phi) = ((2 + \cos(\phi)) \cos(\theta), (2 + \cos(\phi)) \sin(\theta), \sin(\phi))$.

5) (cylindrical and spherical coordinates) a) What is the equation for the surface $x^2 + y^2 - 5x = z^2$ in cylindrical coordinates?
b) Describe in words or draw a sketch of the surface whose equation is $\rho = \sin(3\phi)$ in spherical coordinates (ρ, θ, ϕ).

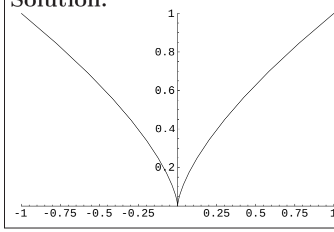
Solution:
a) $r^2 - 5r\cos(\theta) = z^2$.
b) Draw the surface first in the rz plane. Here you see the picture.



Section 2.3: Parametrized Curves

1) (parametrized curves) Sketch the plane curve $\vec{r}(t) = \langle x(t), y(t)\rangle = \langle t^3, t^2\rangle$ for $t \in [-1, 1]$ by plotting the points for different values of t. Calculate its velocity $\vec{r}\,'(t)$ as well as its acceleration $\vec{r}\,''(t)$ at the point $t = 2$.

Solution:



The velocity is $\vec{r}\,'(t) = (3t^2, 2t)$. The acceleration is $\vec{r}\,''(t) = (6t, 2)$. At the time $t = 2$ we have $\vec{r}\,'(2) = (12, 4)$ and $\vec{r}\,''(2) = (12, 2)$.

2) (reconstructing curves from acceleration) A device in a car measures the acceleration $\vec{r}\,''(t) = \langle\cos(t), -\cos(3t)\rangle$ at time t. Assume that the car is at the origin $(0, 0)$ at time $t = 0$ and has zero speed at $t = 0$, what is its position $\vec{r}(t)$ at time t?

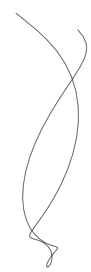
Solution:
$\vec{r}\,'(t) = (\sin(t), -\sin(3t)/3) + (C_1, C_2)$. Because the car has zero speed at time $t = 0$, we have $C_1 = C_2 = 0$.
From $\vec{r}\,'(t) = (\sin(t), -\sin(3t)/3)$, we obtain $\vec{r}(t) = (-\cos(t), +\cos(3t)/9) + (C_1, C_2)$. Because $\vec{r}(0) = (0, 0)$, we have $C_1 = 1, C_2 = -1/9$. $\vec{r}(t) = (-\cos(t) + 1, \cos(3t)/9 - 1/9)$.

3) (curves on surfaces) Verify that the curve $\vec{r}(t) = (t\cos(t), 2t\sin(t), t^2)$ is located on the elliptic paraboloid

$$z = x^2 + y^2/4 .$$

Use this fact to sketch the curve.

Solution:
Just plug in $x(t)^2 + y(t)^2 = z$.



4) (intersections of surfaces) Find the parameterization $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ of the curve obtained by intersecting the elliptical $x^2/9 + y^2/4 = 1$ with the surface $z = xy$. Find the velocity vector $\vec{r}\,'(t)$.

Solution:
We find first $x(t) = 3\cos(t), y(t) = 2\sin(t)$ using the first equation. Then get $z(t) = x(t)y(t) = 6\cos(t)\sin(t)$.
$\vec{r}(t) = (x(t), y(t), z(t)) = (3\cos(t), 2\sin(t), 6\cos(t)\sin(t)/2)$. The velocity vector is $\vec{r}'(t) = (x'(t), y'(t), z'(t)) = (-3sin(t), 2\cos(t), 6\cos^2(t) - 6\sin^2(t))$.

5) (parametrized curves) Consider the curve $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle = \langle t^2, 1+t, 1+t^3\rangle$. Check that it passes through the point $(1, 0, 0)$ and find the velocity vector $\vec{r}\,'(t)$, the acceleration vector $\vec{r}\,''(t)$ as well as the jerk vector $\vec{r}\,'''(t)$ at this point.

Solution:
The curve passes through the point at $t = -1$.
$\vec{r}\,'(t) = (2t, 1, 3t^2)$,
$\vec{r}\,''(t) = (2, 0, 6t)$,
$\vec{r}\,'''(t) = (0, 0, 6)$
At time $t = -1$, we obtain $\vec{r}\,'(-1) = (-2, 1, 3)$,
$\vec{r}\,''(-1) = (2, 0, -6)$,
$\vec{r}\,'''(-1) = (0, 0, 6)$

Section 2.4: Arc length and curvature

1) (arc length) Find the arc length of the curve $\vec{r}(t) = \langle t^2, \sin(t) - t\cos(t), \cos(t) + t\sin(t)\rangle$, $0 \le t \le \pi$.

Solution:
The velocity is $\vec{r}\,'(t) = \langle 2t, t\sin(t), t\cos(t)\rangle$ and the speed is $|\vec{r}\,'(t)|\sqrt{5t^2} = \sqrt{5}t$. The arc length of the curve is $\int_0^\pi \sqrt{5}t\;dt = \pi^2\sqrt{5}/2$.

2) (curvature) Find the curvature of $\vec{r}(t) = \langle e^t\cos(t), e^t\sin(t), t\rangle$ at the point $(1, 0, 0)$.

Solution:
To use the formula $\kappa(t) = |\vec{r}\ '(t) \times \vec{r}\ ''(t)|/|\vec{r}\ '(t)|^3$, we need to know the velocity $\vec{r}\ '(t) = (e^t(\cos(t) - \sin(t)), e^t(\sin(t) + \cos(t)), 1)$, as well as the acceleration $\vec{r}\ ''(t) = (-2e^t\sin(t), 2e^t\cos(t), 0)$. At the time $t = 0$, we have $\vec{r}\ '(0) = (1, 1, 1)$ and $\vec{r}\ ''(t) = (0, 2, 0)$. Now apply the formula $\kappa(0) = |(1, 1, 1) \times (0, 2, 0)|/\sqrt{3}^3 = |(-2, 0, 2)|/\sqrt{3}^3 = \sqrt{8}/\sqrt{3}^3 = 2\sqrt{6}/9$.

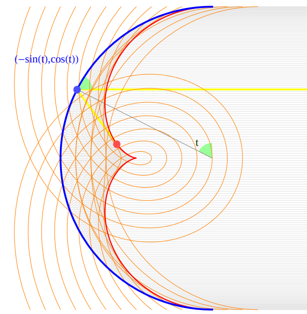
3) (unit tangent, normal and binormal vectors) Find the vectors $\vec{T}(t), \vec{N}(t)$ and $\vec{B}(t)$) for the curve $\vec{r}(t) = \langle t^2, t^3, 0\rangle$ for $t = 2$.
Do the vectors depend continuously on t for all t?

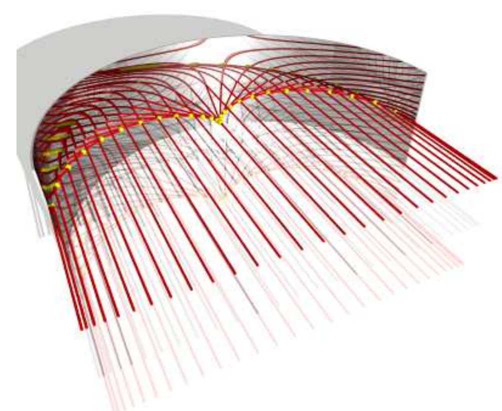
Solution:
$\vec{T}(t) = \vec{r}\ '(t)/|\vec{r}\ '(t)| = \langle 1, 3, 0\rangle/\sqrt{10}$, $\vec{N}(t) = \vec{T}\ '(t)/|\vec{N}\ '(t)| = \langle -3, 1, 0\rangle/\sqrt{10}$, $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \langle 0, 0, 1\rangle$.
The $\vec{T}$ and $\vec{N}$ vectors do depend continuously on t. While $\vec{T}(t) = \langle 2t, 3t^2, 0\rangle/\sqrt{4t^2 + 9t^4}$ looks discontinues at $t = 0$ at first, one can divide that formula by t to get $\vec{T}(t) = \langle 2, 3t, 0\rangle/\sqrt{4 + 9t^2}$ which is smooth in t. Also the second derivative $T'(t)$ is smooth. The third vector, as a cross product depends continuously on t also.

4) (curvature) Let $\vec{r}(t) = \langle t, t^2\rangle$. Find the equation for the **caustic** $\vec{s}(t) = \vec{r}(t) + \vec{N}(t)/\kappa(t)$ known also as the **evolute** of the curve.

Solution:
The curvature of the graph of $f(x) = x^2$ is $\kappa(t) = f''(t)/(1 + f'(t)^2)^{3/2} = 2/(1 + 4t^2)^{3/2}$. The normal vector to the curve is $\vec{n}(t) = \langle -2t, 1\rangle$ which is orthogonal to the velocity vector $\vec{v}(t) = \langle 1, 2t\rangle$. The unit normal vector is $\vec{N}(t) = \langle -2t, 1\rangle/\sqrt{4t^2 + 1}$. The caustic is $\vec{s}(t) = \vec{r}(t) + \frac{1}{2}(1 + 4t^2)^{3/2}(-2t, 1)/\sqrt{4t^2 + 1} = \boxed{(4x^3, 1/2 - 3t^2)}$.

5) (curvature and curves) If $\vec{r}(t) = \langle -\sin(t), \cos(t)\rangle$ is the boundary of a coffee cup and light enters in the direction $\langle -1, 0\rangle$, then light focuses inside the cup on a curve which is called the **coffee cup caustic**. The light ray travels after the reflection for length $\sin(\theta)/(2\kappa)$ until it reaches the caustic. Find a parameterization of the caustic.





Solution:
If $\vec{r}(t) = \langle -\sin(t), \cos(t) \rangle$ is the boundary of the cup and light enters in the direction $(-1, 0)$, then the impact angle θ is just t. The curvature $\kappa(t)$ is 1. Parallel light coming from the right focuses at infinity. The light which leaves into the direction $(\cos(2t), \sin(2t))$ focuses after reflection at a distance $e = \sin(\theta)/(2\kappa) = \sin(\theta)/2$. The caustic is therefore parameterized by $\vec{R}(t) = \langle -\sin(t), \cos(t) \rangle - \langle \cos(2t), \sin(2t) \rangle \sin(t)/2 \rangle = \langle -\sin(t) + \cos(2t)\sin(t)/2, \cos(t) + \sin(2t)\sin(t)/2 \rangle$.