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# Homework for Chapter 1. Geometry and Space

## Section 1.1: Space, distance, geometrical objects

1) (Geometrical objects) Describe and sketch the set of points $P = (x, y, z)$ in $\mathbf{R}^3$ represented by

| | |
|---|---|
| a) $9y^2 + 4z^2 = 81$ | e) $xy = 4$ |
| b) $x/7 - y/11 - z/13 = 1$ | f) $x^2 + (y-2)^2 = 0$ |
| c) $(x+3)^2 = 25$ | g) $x^2 - y^2 = 0$ |
| d) $d(P,(1,0,0)) + d(P,(0,1,0) = 10$. | h) $x^2 = y$. |

**Solution:**
a) An elliptical cylinder with the $x$ axis as a center width 3 and height 4.5.
b) A plane through the points $(7,0,0), (0,-11,0), (0,0,-13)$.
c) The union of two planes $x = -8$ and $y = -2$.
d) An ellipsoid with focal points $(1,0,0)$ and $(0,1,0)$.
e) A hyperbolic cylinder.
f) A line $x = 0, y = 2$, an axes parallel to the $z$ axes.
g) A union of two planes which are perpendicular to each other and intersect in the z-axis. The $z-$trace forms 45 degree angles with the $x$ and $y$ axes.
h) A cylindrical paraboloid.

2) (Distances)

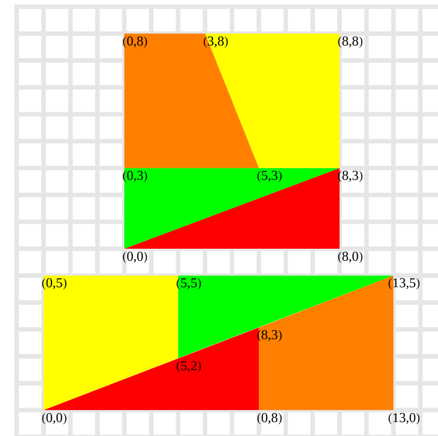
a) Find the distance from the point $P = (3,2,5)$
i) to the $y$-axis.
ii) to the $xz$-coordinate plane.

**Solution:**
(i) A general point $(x,y,z)$ has distance $\sqrt{x^2+z^2}$ from the $y$-axis and distance $y$ from the $xz$-coordinate system. In our case, the point $P$ has distance $\sqrt{9+25}$ from the $y$-axis.
(ii) The distance is 2 from the $xz$-coordinate plane.

3) (Distances)

Below you see two rectangles, one of area $8 \cdot 8 = 64$, an other of area $65 = 13 \cdot 5$. But there are triangles or trapezoids which match. What is going on? Look at distances.






**Solution:**
Not all points which appear to be on lines are really on lines. Look at the second picture. The line from $(0,0)$ to $(5.2)$ to $(8.3)$ and $(13,5)$ is broken. The distances do not add up. For example, $\sqrt{29} + \sqrt{10} = 8.547...$ does not add up to $\sqrt{73} = 8.544....$ If you draw everything right, there will be a narrow piece of area which is left out. This explains the larger area in the second picture.

4) (Spheres, traces) Find an equation of the sphere with center $(-1,8,-5)$ and radius 6. Describe the traces of this surface, its intersection with each of the coordinate planes.

**Solution:**
$(x+1)^2 + (y-8)^2 + (z+5)^2 = 36$ is the equation of the sphere. The traces are obtained by putting $x = 0$ or $y = 0$ or $z = 0$.
$x = 0$ gives the $yz$-trace: $(y-8)^2 + (z+5)^2 = 35$ is a circle.
$y = 0$ gives the $xz$-trace: $(x+1)^2 + (z+5)^2 = 36 - 64$ is the empty set.
$z = 0$ gives the $xy$-trace: $(x+5)^2 + (y-8)^2 = 11$ is a circle

5) (Spheres, distances) Four spheres of equal radius are located in space so that any pair of two spheres touch. The centers of three spheres are known to be $(\sqrt{2},0,0), (0,\sqrt{2},0), (0,0,\sqrt{2})$ and the forth sphere with coordinates $(-a,-a,-a)$ is located in the octant $\{x < 0, y < 0, z < 0\}$. Find the radius of the spheres as well as the center of the fourth sphere.

**Solution:**
The distance between two of the three spheres is 2 so that the radius is 1. The fourth sphere has to have the coordinates $P = (-a,-a,-a)$. In order that this sphere touches the first spheres, we must have that the distance between $P$ and the center of one of the spheres is equal to 2 too. This means

$$(a+\sqrt{2})^2 + a^2 + a^2 = 4 \; .$$

Solving the quadratic equation gives $a = \sqrt{2}/3$.

## Section 1.2: Vectors, dot product, projections

1) (Vector operations) Let $\vec{u} = \langle 2,3 \rangle$ and $\vec{v} = \langle -2,1 \rangle$.
a) Draw the vectors $\vec{u}, \vec{v}, \vec{u}+\vec{v}, \vec{u}-\vec{v}$.
b) What is the relation between the length of $\vec{u}-\vec{v}$ and the lengths of $\vec{u}$ and $\vec{v}$?
c) Prove the Pythagoras theorem: if $\vec{u}, \vec{v}$ are orthogonal $\vec{u}\cdot\vec{v}=0$, then $|\vec{u}-\vec{v}|^2 = |\vec{u}|^2+|\vec{v}|^2$.

**Solution:**
a) draw a parallelogram. $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are the diagonals.
b) $|\vec{u}-\vec{v}| \geq |\vec{u}| - |\vec{v}|$, the triangle inequality.
c) $(\vec{u}-\vec{v}) \cdot (\vec{u}-\vec{v}) = \vec{u}\cdot\vec{u} - 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} = |\vec{u}|^2 + |\vec{v}|^2$, where we used $\vec{u}\cdot\vec{v} = 0$.

2) (Orthogonality) a) Verify that if $s,t$ are two integers, then $\vec{v} = \langle x,y \rangle$ is a vector with integer length if $x = 2st, y = t^2 - s^2$. This gives **Pythagorean triples** $x^2 + y^2 = z^2$.
b) An **Euler brick** is a cuboid of dimensions a,b,c such that the face diagonals are integers. Verify that $\langle a,b,c \rangle = \langle u(4v^2-w^2), v(4u^2-w^2), 4uvw \rangle$ leads to an Euler brick if $u^2+v^2=w^2$.
c) Verify that $\vec{v} = \langle a,b,c \rangle = \langle 240, 117, 44 \rangle$ is a vector which leads to an Euler brick.

P.S. If also the space diagonal $\sqrt{a^2+b^2+c^2}$ is an integer, an Euler brick is called a **perfect cuboid**. It is an open mathematical problem, whether a perfect cuboid exists. Nobody has found one, nor proven that it can not exist.

**Solution:**
a) $(x,y) = (2st, t^2-s^2)$ has length $\sqrt{4s^2t^2+t^4+s^4-2s^2t^2} = (t^2+s^2)$.
b) We have to verify that $a^2+b^2$ and $b^2+c^2$ and $a^2+c^2$ are all squares

$$\begin{aligned} a^2+b^2 &= u^2(4v^2-w^2)^2 + v^2(4u^2-w^2)^2 = u^6+3u^4v^2+3u^2v^4+v^6 = (u^2+v^2)^3 = w^6 = (w^3)^2 \\ b^2+c^2 &= v^2(4u^2-w^2)^2 + (4uvw)^2 = v^2(16u^4-8u^2w^2+w^4-16u^2w^2) = v^2(4u^2+w^2)^2 \\ a^2+c^2 &= u^2(4v^2-w^2)^2 + (4uvw)^2 = u^2(16v^4-8v^2w^2+w^4-16v^2w^2) = u^2(4v^2+w^2)^2 \end{aligned}$$

c) $240^2+117^2 = 267^2$, $117^2+44^2 = 125^2$, $240^2+117^2 = 244^2$.

3) (Angles and projection) **Colors** are encoded by vectors $\vec{v} = (r,g,b)$, where the **red**, **green** and **blue** components are all numbers in the interval $[0,1]$. Examples are:

| | | | |
|---|---|---|---|
| (0,0,0) | black | (0,0,1) | blue |
| (1,1,1) | white | (1,1,0) | yellow |
| (1/2,1/2,1/2) | gray | (1,0,1) | magenta |
| (1,0,0) | red | (0,1,1) | cyan |
| (0,1,0) | green | (1,1/2,0) | orange |
| (0,1,1/2) | spring green | (1,1,1/2) | khaki |
| (1,1/2,1/2) | pink | (1/2,1/4,0) | brown |

a) Determine the angle between the colors magenta and cyan.
b) Find a color which is both orthogonal to orange and yellow.
c) What does the scaling $\vec{v} \mapsto \vec{v}/2$ do, if $\vec{v}$ represents a color?
d) If $\vec{v}$ and $\vec{w}$ are two colors, their mixture $(\vec{v}+\vec{w})/2$ is also a color. What is the mixture of red and white?
e) Vectors on the diagonal $r = g = b$ are called **gray** colors. Find the gray vector which is the vector projection of yellow onto white.

**Solution:**
a) $\cos(\alpha) = \frac{(1,0,1)\cdot(0,1,1)}{(|(1,0,1)||(0,1,1)|)} = 1/2$ gives $\alpha = 60^\circ = \pi/3$.
b) Blue $\vec{b} = \langle 0,0,1 \rangle$ is orthogonal to orange $\vec{y} = \langle 1, 1/2, 0 \rangle$ and yellow because the dot product vanishes.
c) It darkens the color.
d) Pink.
e) The vector projection of yellow $\vec{y} = \langle 1,1,0 \rangle$ onto white $\vec{w} = \langle 1,1,1 \rangle$ is $\vec{w}(\vec{y}\cdot\vec{w})/|\vec{w}|^2 = \langle 2/3, 2/3, 2/3 \rangle$.

4) (Angles) a) Find the angle between the diagonal of the unit cube and one of the diagonal of one of its faces. Assume that the two diagonals go through the same edge of the cube. Remark. You can leave the answer in the form $\cos(\alpha) = ....$
b) Find the angle between two face diagonals which go through the same edge and are on adjacent faces.

**Solution:**
a) Use a coordinate system so that

$$(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1), (1,1,1)$$

are the corners of the cube. The main diagonal is the vector $\vec{v} = (1,1,1)$, a diagonal in the face is $\langle 1,1,0 \rangle$. Then, $\cos(\alpha) = 2/(\sqrt{3}\sqrt{2})$.
b) We can take the two diagonals $(1,1,0), (0,1,1)$ and the angle satisfies $\cos(\alpha) = 1/2$ so that $\alpha = \pi/3$.

5) (Length, angle and projection) Assume $\vec{v} = \langle -4,2,2 \rangle$ and $\vec{w} = \langle 3,0,4 \rangle$.

a) Find the length of $\vec{w}$ and the dot product between $\vec{v}$ and $\vec{w}$.
b) Find the vector projection of $\vec{v}$ onto $\vec{w}$.
c) Find the component of $\vec{v}$ on $\vec{w}$.
d) Find a vector parallel to $\vec{w}$ of length 1.

**Solution:**
a) $|\vec{w}| = 5$, $\vec{v}\cdot\vec{w} = -4$.
b) $\vec{w}(\vec{v}\cdot\vec{w})/|\vec{w}|^2 = -4/25(3,0,4) = (-12/25, 0, -16/25)$.
c) $(\vec{v}\cdot\vec{w})/|\vec{w}| = -4/5$.
d) $\vec{w}/|\vec{w}| = (3/5, 0, 4/5)$.

## Section 1.3: The cross product and triple scalar product

1) (Cross product)
a) Find the cross product $\vec{w}$ of $\vec{u} = (-3,-1,2)$ and $\vec{v} = (-2,-2,5)$.
b) Find a unit vector $\vec{n}$ orthogonal to $\vec{u}$ and $\vec{v}$.
c) Find the volume of the parallelepiped spanned by $\vec{u}, \vec{v}$ and $\vec{w}$.

**Solution:**
a) $\vec{u} \times \vec{v} = \vec{w} = (-1, 11, 4)$
b) $|\vec{w}| = \sqrt{138}$, $\vec{n} = \vec{w}/|\vec{w}| = (-1/\sqrt{138}, 11/\sqrt{138}, 4/\sqrt{138})$.
c) $\vec{w} \cdot (\vec{u} \times \vec{v}) = \vec{w} \cdot \vec{w} = |\vec{w}|^2 = 138$.

2) (Distance formulas) a) Find the distance between the point $P = (2, -1, 2)$ and the line $x = y = z$.
b) Find a parametrization $r(t)$ of the line given in $a$) and find the minimum of the function $f(t) = d(P, r(t))$. Check whether the minimal value is the distance you got in a).

3) (Area formula) a) Assume $\vec{u} + \vec{v} + \vec{w} = \vec{0}$. Verify that $\vec{u} \times \vec{v} = \vec{v} \times \vec{w} = \vec{w} \times \vec{u}$.
b) Find $(\vec{u} + \vec{v}) \cdot (\vec{v} \times \vec{w})$ if $\vec{u}, \vec{v}, \vec{w}$ are unit vectors which are orthogonal to each other and $\vec{u} \times \vec{v} = \vec{w}$.
c) Find the area of a triangle ABC in the plane if all three points $A, B, C$ have integer coordinates.

**Solution:**
a) Build a triangle with the three vectors $u, v, w$. Each of the terms in the identity is twice the area of the triangle.
b) We have $(\vec{u} + \vec{v}) \cdot (\vec{v} \times \vec{w}) = \vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{w} \cdot (\vec{u} \times (\vec{v}) = \vec{w} \cdot \vec{w} = 1$.
c) The formula for the area is $|(A - C) \times (B - C)|/2$.

4) (Lines and planes) a) Find the parametric and symmetric equation for the line which passes through the points $P = (1, 2, 3)$ and $Q = (3, 4, 5)$.
b) Find the equation for the plane which contains the three points $P = (1, 2, 3), Q = (3, 4, 4)$ and $R = (1, 1, 2)$. This problem is a preparation for next week. You can use without proof that a plane has the form $ax + by + cz = d$ where $\vec{n} = \langle a, b, c\rangle$ is thew normal vector to the plane.

**Solution:**
a) The vector $\vec{v} = \langle 2, 2, 2\rangle$ connects the two points. The parametric equation is $P + t\vec{v} = \langle 1, 2, 3\rangle + t\langle 2, 2, 2\rangle = \langle 1 + 2t, 2 + 2t, 3 + 2t\rangle$. The symmetric equation is $(x - 1)/2 = (y - 2)/2 = (z - 3)/2$.
b) A normal vector $\vec{n} = \langle 1, -2, 2\rangle = \langle a, b, c\rangle$ of the plane $ax + by + cz = d$ is obtained as the cross product of $P - Q$ and $R - Q$ With $d = \vec{n} \cdot P = 3$, we have the equation $x - 2y + 2z = 3$.

5) (Triple scalar product) Verify that the volume of a tetrahedron with corners $P, Q, R, S$ is the absolute value of $\vec{PS} \cdot |\vec{PQ} \times \vec{PR}|/6$.

**Hint.** Find first a formula for the area of one of its triangular faces, and then a formula for the distance from the fourth point to that face.

**Solution:**
The triangle $PQR$ has the area $|n|/2 = |PQ \times PR|/2$. The distance of $S$ to the triangle is $PS \times n/|n|$ so that the volume is $|PS \times n|/6 = |(PQ \times PR) \cdot PS|/6$ which is 1/6 of the volume of the parallelepiped spanned by the sides of the tetrahedron.

## Section 1.4: Lines, planes and distances



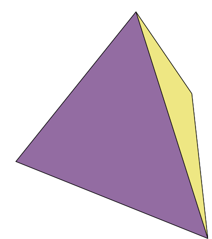
1) (Planes) Find the equation for the plane which contains the point $P = (1, 2, 3)$ and the line which passes through $Q = (3, 4, 4)$ and $R = (1, 1, 2)$.

**Solution:**
A normal vector $\vec{n} = (1, -2, 2) = (a, b, c)$ of the plane $ax + by + cz = d$ is obtained as the cross product of $P - Q$ and $R - Q$ With $d = \vec{n} \cdot P = 3$, we have the equation $x - 2y + 2z = 3$.

2) (Distance formula) Find the distance between the sphere of radius 1 centered at $(2, -1, 2)$ and the plane $4x - 2y + z = 2$.

**Solution:**
The point $Q = (0, 0, 2)$ is on the plane. The scalar projection of $P - Q = (2, -1, 0)$ onto the normal vector $(4, -2, 1)$ of the plane is $10/\sqrt{21}$. The distance of the sphere is 1 less.

3) (Distance formula) A regular tetrahedron has vertices at the points $P_1 = (0, 0, 3), P_2 = (0, \sqrt{8}, -1)$, $P_3 = (-\sqrt{6}, -\sqrt{2}, -1)$ and $P_4 = (\sqrt{6}, -\sqrt{2}, -1)$. Find the distance between two edges which do not intersect.



**Solution:**
The vector $\vec{v} = (P_2 - P_1) = (0, 2\sqrt{2}, -4)$ is parallel to the first edge, the vector $\vec{w} = (P_4 - P_3) = (2\sqrt{6}, 0, 0)$ is parallel to the second edge. The cross product of $\vec{v}$ and $\vec{w}$ is $\vec{n} = (0, -8\sqrt{6}, -8\sqrt{3})$. The distance between the two edges is the scalar projection of $P_3 - P_1$ onto $\vec{n}$. It is $(P_3 - P_1).\vec{n}/|\vec{n}| = 2\sqrt{3}$.

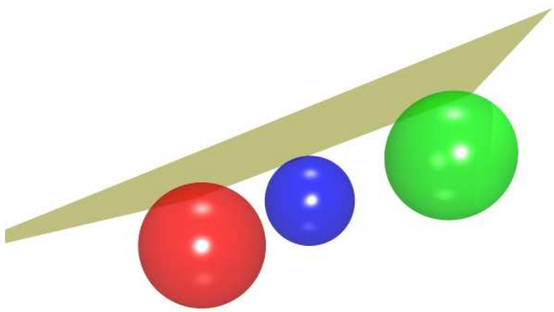
4) (Geometric constructions) Find a parametric equation for the line through the point $P = (3, 1, 2)$ that is perpendicular to the line $L : x = 1 + t, y = 1 - t, z = 2t$ and intersects this line in a point $Q$.

**Solution:**
The point $Q = (1, 1, 0)$ is on the line. The vector $\vec{v} = (1, -1, 2)$ parallel to the line. We have $P - Q = (2, 0, 2)$. The vector $\vec{n} = \vec{v} \times (\vec{v} \times (P - Q)) = (-6, -6, 0)$ is the direction from $P$ to the normal intersection with the line. The line can be given by $\vec{r}(t) = (3 - 6t, 1 - 6t, 2)$.

**Solution:**
The solution is $\sqrt{2}$. You can get the answer using the distance formula $|(P-Q)\times v|/|v|$.

5) (Geometric constructions) Given three spheres of radius 1 centered at $A=(1,2,0), B=(4,5,0), C=(1,3,2)$. Find a plane $ax+by+cz=d$ which touches each of three spheres from the same side.

**Hint.** There are two such planes. You want to consider the plane through the three points $A,B,C$ first. You only need to find one of the two possible planes touching all the spheres on the same side.



**Solution:**
The normal vector to the plane is $\vec{n}=\langle 3,3,0\rangle \times \langle 0,1,2\rangle = \langle 6,-6,3\rangle$. The plane touching the three spheres has the equation $6x-6y+3z=d$, where $d$ is a constant still to be determined. To find this constant, we have to find a point $P$ on the plane. We do that by going from the point $A$ by 1 unit in the direction of the normal vector. The point $P=A+\vec{n}/|\vec{n}| = (1,2,0)+\langle 6/9,-6/9,3/9\rangle = (5,4,1)/3$ is on the plane. Plug in this point into the equation gives $d=3$. The equation of the plane is $\boxed{6x-6y+3z-3=0}$.