8/6/2009 FINAL EXAM PRACTICE IV **Maths 21a, O. Knill, Summer 2009**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) (20 points)

1) T **F** The quadratic surface $-x^2 + y^2 + z^2 = -5$ is a one-sheeted hyperboloid.

Solution:
It is a two sheeted hyperboloid.

2) **T** F $|\vec{u}\times\vec{v}| < |\vec{u}\cdot\vec{v}|$ implies that the angle α between u and v satisfies $|\alpha| < \pi/4$.

Solution:
Indeed, the condition means that $|\tan(\alpha)| < 1$ which implies that the angle is smaller than $\pi/4$.

3) T **F** $\int_0^3 \int_0^{2\pi} r\sin(\theta)\ d\theta\ dr$ is the area of a disc of radius 3.

Solution:
There is a factor $\sin(\theta)$ too much.

4) **T** F If a vector field $\vec{F}(x,y,z)$ satisfies $\text{curl}(F)(x,y,z) = 0$ for all points (x,y,z) in space, then $\vec{F}$ is a gradient field.

Solution:
True. We have derived this from Stokes theorem.

5) T **F** The acceleration of a parameterized curve $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ is zero if the curve $\vec{r}(t)$ is a line.

Solution:
The acceleration can be parallel to the velocity. An acceleration parallel to the velocity produces a line.

6) T **F** The curvature of the curve $\vec{r}(t) = \langle 3+\sin(t), t, t^2\rangle$ is half of the curvature of the curve $\vec{s}(t) = \langle 6+2\sin(t), 2t, 2t^2\rangle$.

Solution:
If we scale a curve by a factor λ, the curvature gets scaled by $1/\lambda$. The curvature gets smaller if we scale, not larger.

7) | T | | F | The curve $\vec{r}(t) = \langle \sin(t), t, \cos(t) \rangle$ for $t \in [0, \pi]$ is a helix.

Solution:
True. Indeed, one can check that $x(t)^2 + z(t)^2 = 1$.

8) | T | | F | If a function $u(t, x)$ is a solution of the partial differential equation $u_{tx} = 0$, then it is constant.

Solution:
No, any function $u(t, x) = at + bx$ is also a solution too and this solution is not constant.

9) | T | | F | The unit tangent vector $\vec{T}$ of a curve is always perpendicular to the acceleration vector.

Solution:
It is perpendicular to the normal vector $\vec{N}$.

10) | T | | F | Let (x_0, y_0) be the maximum of $f(x, y)$ under the constraint $g(x, y) = 1$. Then the gradient of g at (x_0, y_0) is perpendicular to the gradient of f at (x_0, y_0).

Solution:
It is parallel

11) | T | | F | At a critical point for which $f_{xx} > 0$, the discriminant D determines whether the point is a local maximum or a local minimum.

Solution:
No, it is still possible that $D = 0$.

12) [T] [F]

If a vector field $\vec{F}(x,y)$ is a gradient field, then for any closed curve C the line integral $\int_C \vec{F} \cdot \vec{dr}$ is zero.

Solution:
A very basic fact.

13) [T] [F]

If C is part of a level curve of a function $f(x,y)$ and $\vec{F} = \langle f_x, f_y \rangle$ is the gradient field of f, then $\int_C \vec{F} \cdot \vec{dr} = 0$.

Solution:
The gradient field is perpendicular to the level curves.

14) [T] [F]

The gradient of the divergence of a vector field $\vec{F}(x,y,z) = \nabla f(x,y,z)$ is always the zero vector field.

Solution:
The gradient of the divergence is not zero in general. Take $F(x,y,z) = \langle x^2, 0, 0 \rangle$ for example.

15) [T] [F]

The line integral of the vector field $\vec{F}(x,y,z) = \langle x, y, z \rangle$ along a line segment from $(0,0,0)$ to $(1,1,1)$ is 1.

Solution:
By the fundamental theorem of line integrals, we can take the difference of the potential $f(x,y,z) = x^2/2 + y^2/2 + z^2/2$, which is $1/2 + 1/2 + 1/2 = 3/2$.

16) [T] [F]

If $\vec{F}(x,y) = \langle x^2 - y, x \rangle$ and $C : \vec{r}(t) = \langle \sqrt{\cos(t)}, \sqrt{\sin(t)} \rangle$ parameterizes the boundary of the region $R : x^4 + y^4 \leq 1$, then $\int_C \vec{F} \cdot \ \vec{dr}$ is twice the area of R.

Solution:
This is a direct consequence of Green's theorem and the fact that the two-dimensional curl $Q_x - P_y$ of $\vec{F} = \langle P, Q \rangle$ is equal to 2.

17) [T] [F]

The flux of the vector field $\vec{F}(x,y,z) = \langle 0, 0, z \rangle$ through the boundary S of a solid torus E is equal to the volume the torus.

Solution:
It is the **volume** of the solid torus.

18) | T | F |

If $\vec{F}$ is a vector field in space and S is the boundary of a cube then the flux of $\text{curl}(\vec{F})$ through S is 0.

Solution:
This is true by Stokes theorem.

19) | T | F |

If $\text{div}(\vec{F})(x, y, z) = 0$ for all (x, y, z) and S is a half sphere then the flux of $\vec{F}$ through S is zero.

Solution:
It would be consequence of the divergence theorem if the surface were closed, but it is not.

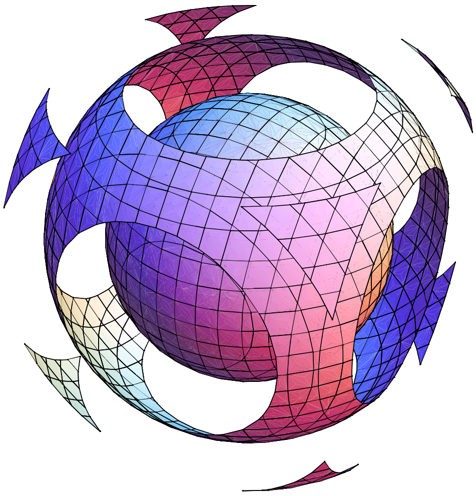
20) | T | F |

If the curl of a vector field is zero everywhere, then its divergence is zero everywhere too.

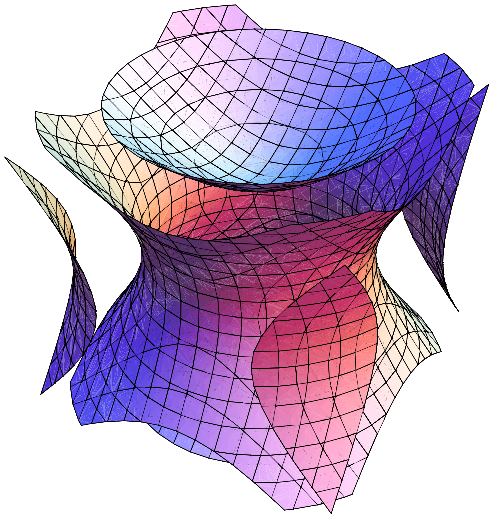
Solution:
Take a gradient field $F(x, y, z) = \langle x, y, z \rangle$. Its curl is zero but its divergence is not.

Problem 2) (10 points)

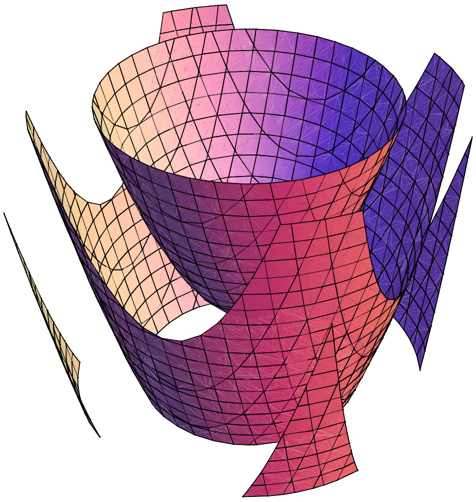
a) Match the following contour surface maps with the functions $f(x, y, z)$



I



II



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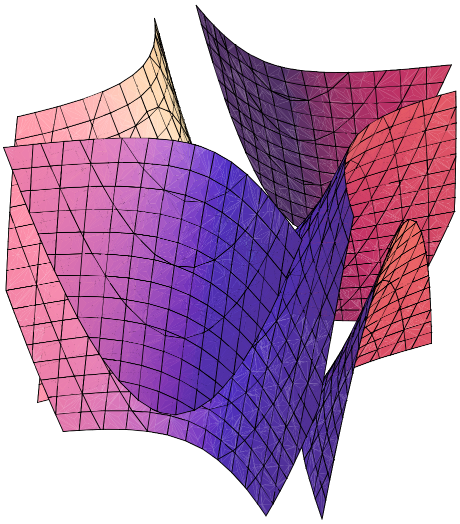
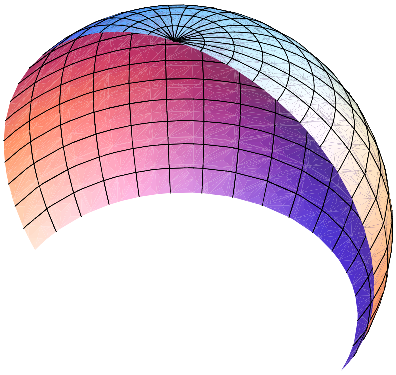


IV

Enter I,II,III,IV here	Function
	$f(x, y, z) = -x^2 + y^2 + z$
	$f(x, y, z) = x^2 + y^2 + z^2$
	$f(x, y, z) = -x^2 - y^2 + z$
	$f(x, y, z) = -x^2 - y^2 + z^2$

b) Match the following parametrized surfaces with their definitions



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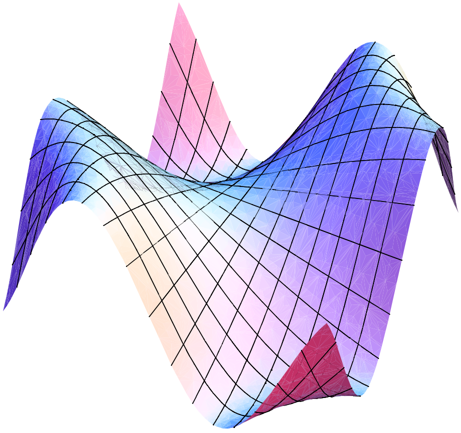

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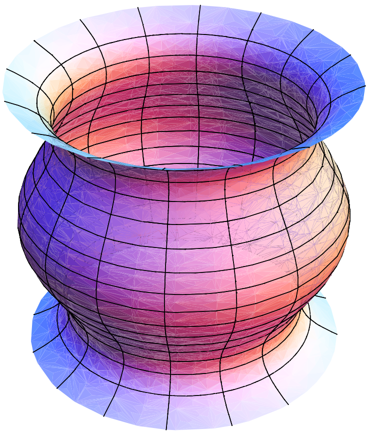

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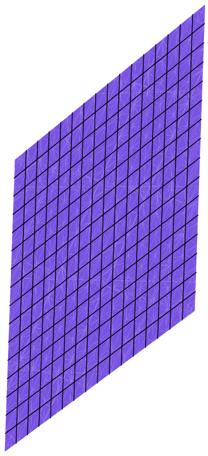


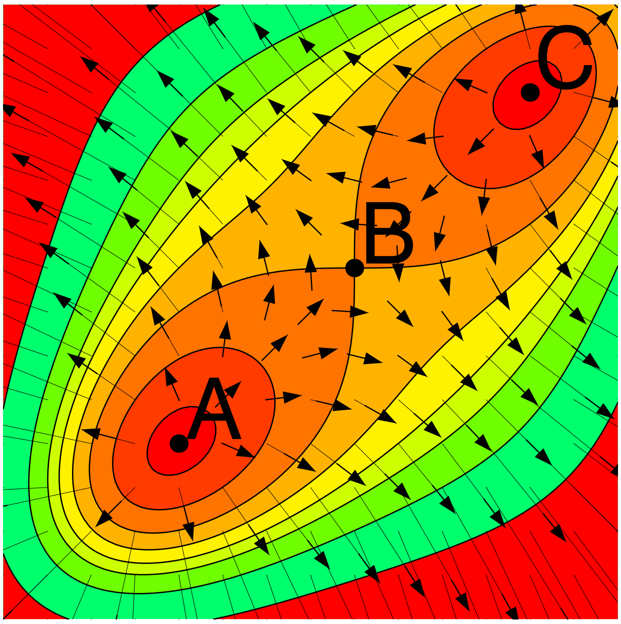
IV

Enter I,II,III,IV here	Function
	$\vec{r}(u,v) = \langle u - v, u + 2v, 2u + 3v\sin(uv)\rangle$
	$\vec{r}(u,v) = \langle \cos(u)\sin(v), 4\sin(u)\sin(v), 3\cos(v)\rangle$
	$\vec{r}(u,v) = \langle (v^4 - v^2 + 1)\cos(u), (v^4 - v^2 + 1)\sin(u), v\rangle$
	$\vec{r}(u,v) = \langle u, v, \sin(uv)\rangle$

Solution:
a) IV,I,III,II
b) II,I,III,IV

Problem 3) (10 points)

No justifications are required in this problem. The first picture shows a gradient vector field $\vec{F}(x,y) = \nabla f(x,y)$.



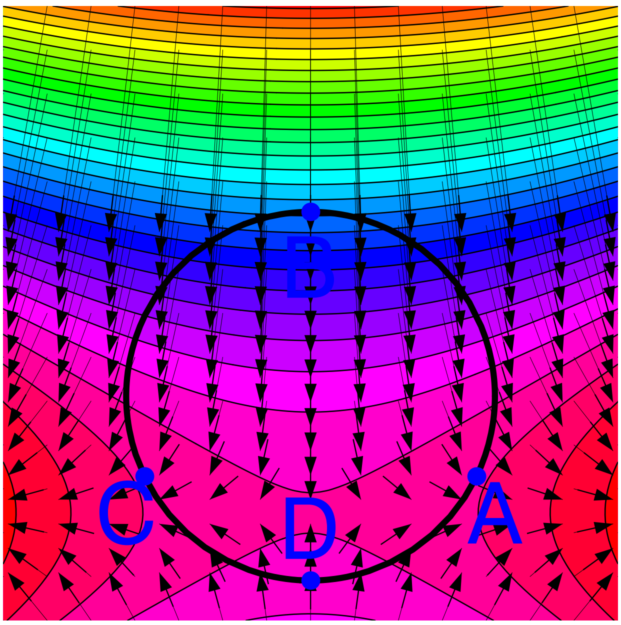
The critical points of $f(x,y)$ are called A, B and C. What can you say about the nature of these three critical points? Which one is a local max, which a local min, which a saddle.

point	local max	local min	saddle
A			
B			
C			

$\int_{P\to Q} \vec{F} \cdot d\vec{r}$ denotes the line integral of $\vec{F}$ along a straight line path from P to Q.

statement	True	False
$\int_{A\to B} \vec{F}\, d\vec{r} \geq 0$		
$\int_{A\to C} \vec{F}\, d\vec{r} \geq \int_{A\to B} \vec{F}\, d\vec{r}$		

The second picture again shows an other gradient vector field $\vec{F} = \nabla f(x,y)$ of a different function $f(x,y)$.



We want to identify the maximum of $f(x,y)$ subject to the constraint $g(x,y) = x^2 + y^2 = 1$. The solutions of the Lagrange equations in this case are labeled A, B, C, D. At which point on the circle if f maximal?

point	maximum
A	
B	
C	
D	

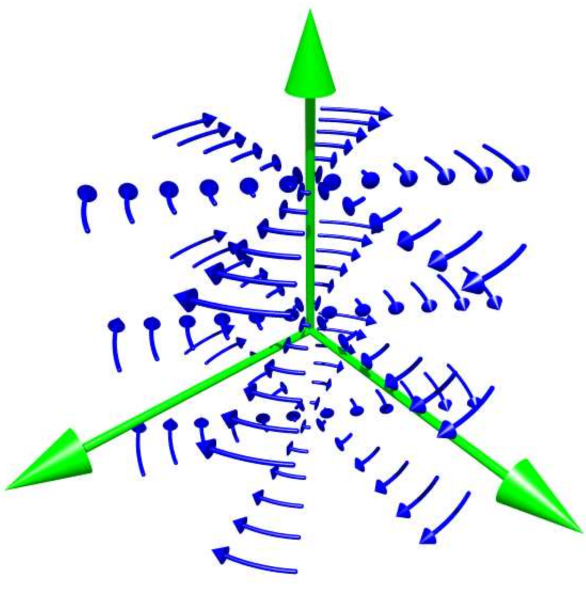
$\int_\gamma \vec{F} \cdot d\vec{r}$ denotes the line integral of $\vec{F}$ along the circle $\gamma : x^2 + y^2 = 1$, oriented counter clockwise.

$\int_\gamma \vec{F} \cdot \vec{r}$	> 0	< 0	$= 0$
Check if true:			

Solution:
It is important to realize in this problem that the gradient vector points into the direction where f increases. The points A, C are local minima, the point B is a saddle point. By the **fundamental theorem of line integrals**, the line integral of any path from A to B is $f(B) - f(A)$. This is ≥ 0. The line integral from A to C is $f(C) - f(A)$. This is smaller than the line integral from A to B. The point B is the global minimum of f in the second picture. The line integral of $\vec{F}$ along the circle is zero again by the fundamental theorem of line integrals. A gradient vector field has the property that the line integral along a closed loop is zero.

Problem 4) (10 points)

a) Match the following vector fields in space with the corresponding formulas:



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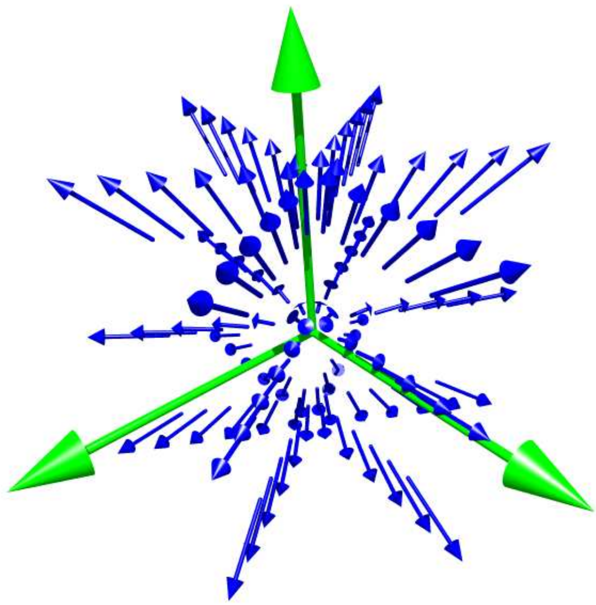
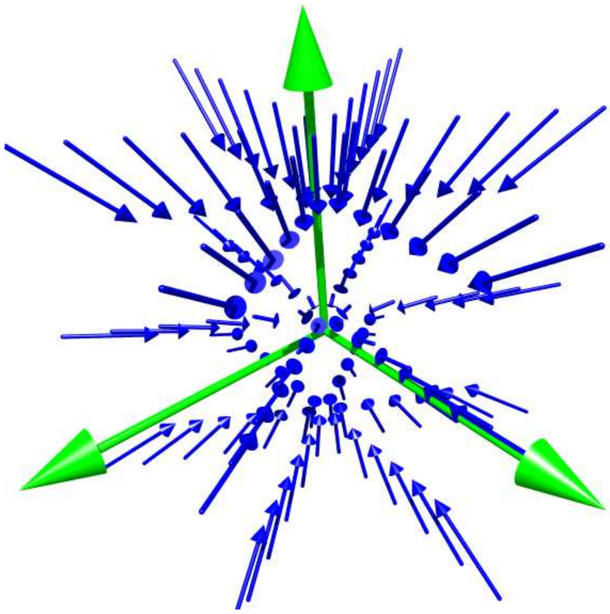
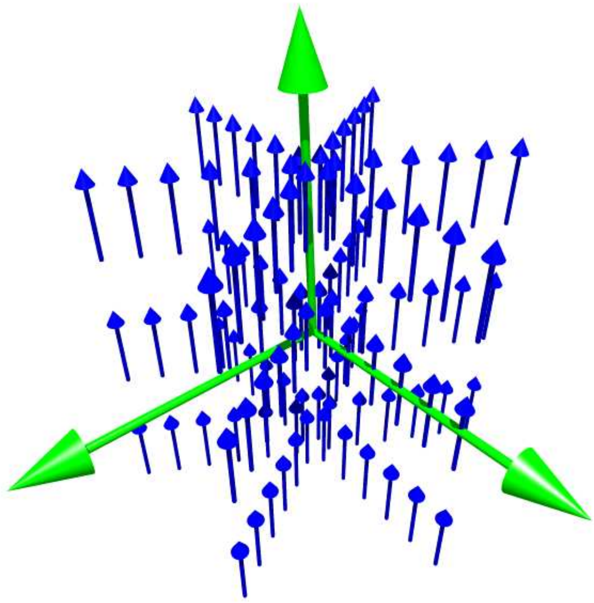


II



III



IV

Enter I,II,III,IV here	Vector Field
	$\vec{F}(x,y,z) = \langle y, -x, 0\rangle$
	$\vec{F}(x,y,z) = \langle x, y, z\rangle$
	$\vec{F}(x,y,z) = \langle 0, 0, 1\rangle$
	$\vec{F}(x,y,z) = \langle -x, -2y, -z\rangle$

b) Choose from the following words to complete the following table: "arc length formula", "surface area formula", "chain rule", "volume of parallel epiped", "area of parallelogram", "Consequence of Clairot theorem", "Fubini Theorem", "line integral", "Flux integral", "vector projection", "scalar projection", "partial derivative":

Formula	Name of formula or rule or theorem
$P_{\vec{v}}(\vec{w}) = \vec{w}\frac{\vec{v}\cdot\vec{w}}{\lvert\vec{w}\rvert^2}$	
$\int\int_R \lvert\vec{r}_u \times \vec{r}_v\rvert \, dudv$	
$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$	
$\mathrm{curl}(\mathrm{grad}(f)) = \vec{0}$	
$\int_a^b \lvert\vec{r}\,'(t)\rvert \, dt$	
$\lvert\vec{u} \cdot (\vec{v} \times \vec{w})\rvert$	

Solution:
a) I,II,IV,III
b) projection, area, chain rule, Clairot, arc length, volume.

Problem 5) (10 points)

The point $P = (1, 2, 2)$ is mirrored at the plane which contains the points $A = (0, 0, 0)$, $B = (4, 0, 2)$ and $C = (2, 2, -1)$. The mirror point is called Q.

a) (3 point) Find a normal vector to the plane which has length 1.

b) (4 points) Find the distance of the point P to the plane.

c) (3 points) Find the coordinates of the point Q.

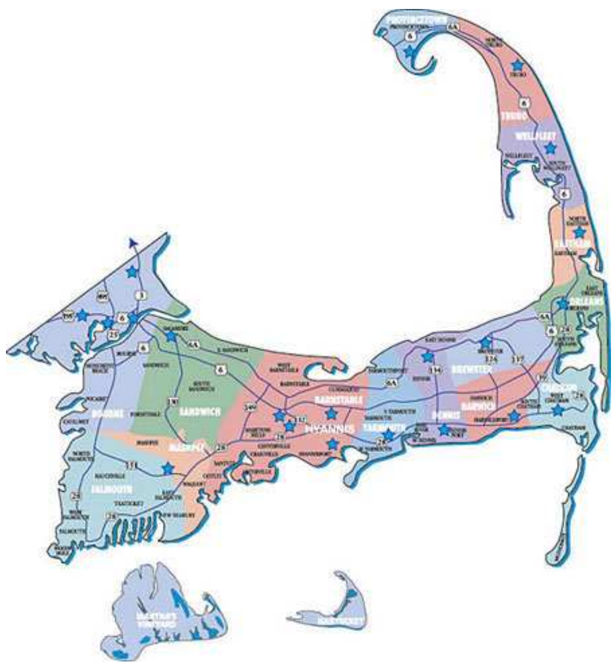
Solution:
a) $\vec{AB} \times \vec{BC} = \langle -4, 8, 8 \rangle$ has length 12 so that $\vec{n} = \langle -4/12, 8/12, 8/12 \rangle = \boxed{\langle -1, 2, 2 \rangle / 3}$ is a normal vector perpendicular to the plane.
b) The distance is $d = |\vec{n} \cdot PA| / |\vec{n}| = |\vec{n} \cdot PA| = \boxed{7/3}$.
c) $P - 2d\vec{n} = (1, 2, 2) - 2 \cdot 7/3 \langle -1, 2, 2 \rangle / 3 = \boxed{(23/9, -10/9, -10/9}$.

Problem 6) (10 points)

The tip of Cape Cod near Provincetown is full of dunes and a nice place to relax. The elevation at position (x, y) is

$$f(x, y) = x^3/12 - xy - 2y - y^2 - x \ .$$

Find all critical points and decide whether they are maxima, minima or saddle points.



Solution:
The gradient is

$$\nabla f = \langle x^2/4 - y - 1, -x - 2 - 2y \rangle \ .$$

Plugging in the second equation into the first gives $x = 0$ or $x = -2$. For $x = 0$ we have $y = -1$. For $x = -2$, we get $y = 0$. The Hessian matrix is

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} \frac{x}{2} & -1 \\ -1 & -2 \end{bmatrix}$$

and the discriminant is $D = det(H) = f_{xx}f_{yy} - f_{xy}^2 = x - 1$.

point	D	f_{xx}	type
$(0, -1)$	-1	0	saddle
$(-2, 0)$	1	-1	max

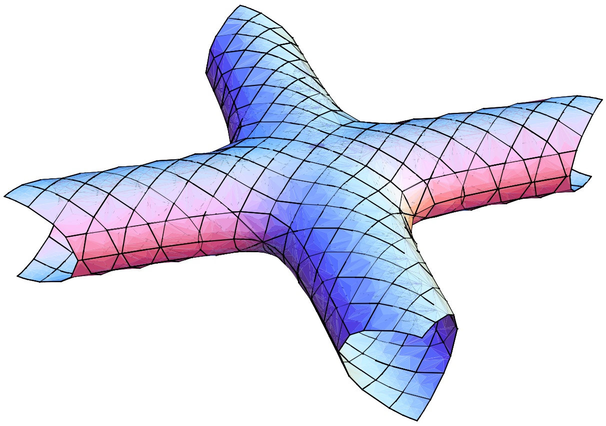
Problem 7) (10 points)

The **roman surface** is defined as the implicit surface

$$f(x, y, z) = x^4 - 2x^2y^2 + y^4 + 4x^2z + 4y^2z + 4x^2z^2 + 4y^2z^2 = 0 \ .$$

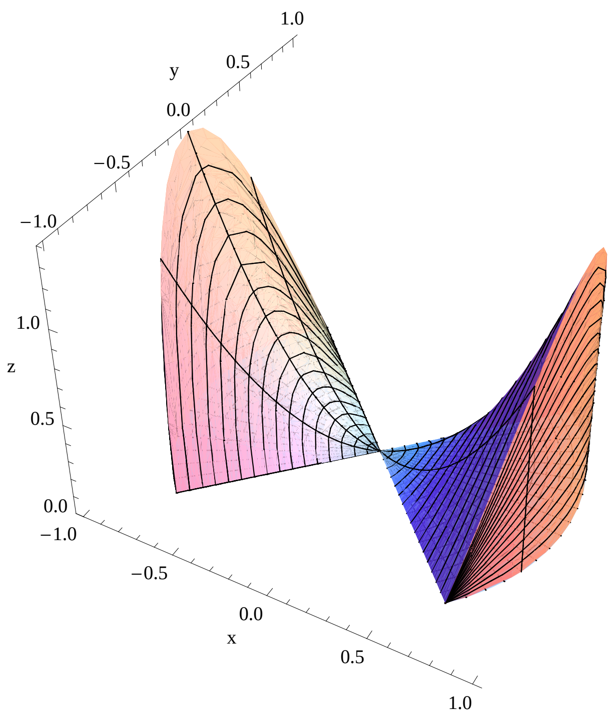
a) (4 points) Find a vector normal to the surface at the point $P = (-1, 1, -1)$.
b) (3 points) Find the equation of the tangent plane at P.
c) (3 points) Estimate $f(-1, 1, -1.1)$ by linear approximation.

Solution:
a) $f_x = 4x^3 - 4xy^2 + 8xz + 8xz^2$, $f_x(-1, 1, -1) = 0$
$f_y = -4x^2y + 4y^3 + 8yz + 8yz^2$, $f_y(-1, 1, -1) = 0$
$f_z = 4x^2 + 4y^2 + 8x^2z + 8y^2$, $f_z(-1, 1, -1) = 8$ The gradient is $\boxed{\langle 0, 0, 8 \rangle}$.
b) The equation is $8z = d$ where d is a constant. Plugging in the point gives $d = -8$. Therefore, the equation is $\boxed{z = -1}$. c) The linearization is $L(x, y, z) = f(-1, 1, -1) + \langle 0, 0, 8 \rangle \cdot \langle x + 1, y - 1, z + 1 \rangle = 0 - 0.8(-0.1) = \boxed{0.8}$.



Problem 8) (10 points)

An excentric architect builts a cinema, which is above the xy-plane and below the surface $z = x^2 - y^2$ and within the solid cylinder $x^2 + y^2 \leq 1$. Find the volume of this building, which features two movie presentation halls.



Solution:
Use cylindrical coordinates and note that the wedge is positive for $-\pi/4 < \theta < \pi/4$ and $\pi - \pi/4 < \theta < \pi + \pi/4$. We compute the volume of half of the solid and double it:

$$2 \int_0^1 \int_{-\pi/4}^{\pi/4} r^2 \cos(2\theta) r \, d\theta dr = 2 \cdot 1/4 = 1/2 \; .$$

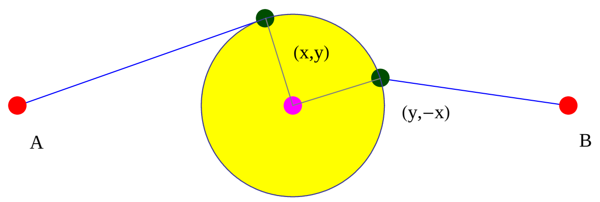
The final result is $\boxed{\frac{1}{2}}$.

Problem 9) (10 points)

A wheel of radius 1 is attached by rubber bands with two points $(-3,0)$ and $(3,0)$. The point (x,y) is attached to $(-3,0)$ and $(y,-x)$ is attached to $(3,0)$. The point (x,y) is constrained to $g(x,y) = x^2 + y^2 = 1$. The wheel will settle at the position, for which the potential energy of the wheel

$$f(x,y) = (x+3)^2 + y^2 + (y-3)^2 + x^2$$

is minimal. Find that position.



Solution:
The Lagrange equations are

$$\begin{aligned} 2(x+3)+2x &= \lambda 2x && (1) \\ 2(y-3)+2y &= \lambda 2y && (2) \\ x^2+y^2 &= 1\ . && (3) \end{aligned}$$

This simplifies to

$$\begin{aligned} 2x+3 &= \lambda x && (4) \\ 2y-3 &= \lambda y && (5) \\ x^2+y^2 &= 1\ . && (6) \end{aligned}$$

Dividing the first by the second gives

$$\begin{aligned} y(2x+3) &= x(2y-3) && (7) \\ x^2+y^2 &= 1\ . && (8) \end{aligned}$$

which gives $x=-y$ and so $x=-y=\pm 1/\sqrt{2}$. Comparing the values $f(-1/\sqrt{2},1/\sqrt{2}) = 20-6\sqrt{2}$ and $f(1/\sqrt{2},-1/\sqrt{2}) = 20+6\sqrt{2}$ shows that $\boxed{(-1/\sqrt{2},1/\sqrt{2})}$ is the minimum.

Problem 10) (10 points)

Find the surface area of the surface

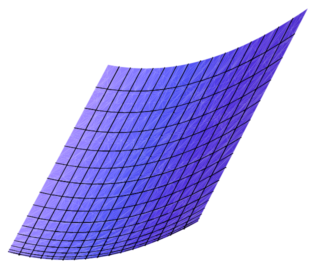
$$\vec{r}(u,v) = \langle u, v^2, \frac{u^2}{\sqrt{2}} + v^2 \rangle\ ,$$

with $0 \leq u \leq 1$ and $0 \leq v \leq 1$.

Hint. In class we have established

$$\int \sqrt{1+x^2}\ dx = x\sqrt{1+x^2}/2 + \text{arcsinh}(x)/2$$

which you can use without having to rederive it.



Solution:
$r_u \times r_v = \langle -2\sqrt{2}uv, -2v, 2v \rangle$. So, we integrate $\int_0^1 \int_0^1 \sqrt{8v^2 + 8u^2v^2} \quad dudv = \frac{\sqrt{8}}{2} \int_0^1 \sqrt{1+u^2}\, du$ which is $\frac{\sqrt{8}}{2}(\frac{\sqrt{2}}{2} + \frac{\text{arcsinh}(1)}{2})$.

Problem 11) (10 points)

Evaluate the double integral

$$\int_0^4 \int_0^{y^2} \frac{x^4}{\sqrt{4-\sqrt{x}}}\, dx\, dy \ .$$

Solution:
Change the order of integration:

$$\int_0^2 \int_{\sqrt{x}}^4 \frac{x^4}{\sqrt{4-\sqrt{x}}}\, dxdy$$

The inner integral can be computed:

$$\int_0^{16} x^4 \sqrt{4-\sqrt{x}}\, dx$$

This is now a solvable integral: make a substitution $u = 4 - \sqrt{x}$, $u = (4-u)^2$ $dx = -2(4-u)$ and get

$$\int_0^4 (4-u)^8 \sqrt{u} 2(4-u)\, du = \int_0^4 (4-u)^9 \sqrt{u}\, du \ .$$

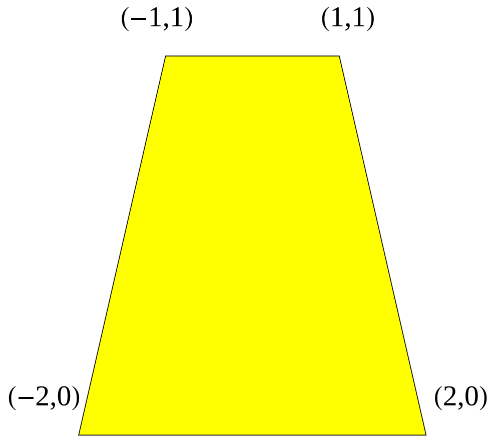
A substitution $v = \sqrt{u}$ gives

$$\int_0^2 (4-v^2)^9 4v^2\, dv \ .$$

Expanding the function as a polynomial leads to a sum of fractions. The result is 549755813888/4849845.
Remark: Since the arithmetic was obviously a bit too heavy, we gave 8 points for a correct reduction to a single interval and 2 points more for a substitution idea.

Problem 12) (10 points)

Find the line integral of the vector field $\vec{F}(x,y) = \langle 3y, 10x + \log(10+\sqrt{y}) \rangle$ along the boundary C of the trapezoid with vertices $(-2,0), (2,0), (1,1), (-1,1)$.



Solution:
This problem already appeared in a practice exam. The curl is 7. By **Greens theorem**, the result is 7 times the area of the region. The area is 3. The final result is $\boxed{21}$.

Problem 13) (10 points)

Let $\vec{F}$ be the vector field

$$\vec{F}(x, y, z) = \langle x + z, y + x, z + y + \cos(\sin(\cos(z)))\rangle$$

and let S be the triangle with vertices $A = (1, 0, 0), B = (0, 1, 0), C = (0, 0, 1)$ parametrized by $\vec{r}(s, t) = \langle 1, 0, 0\rangle + t\langle -1, 1, 0\rangle + s\langle -1, 0, 1\rangle$. Find the line integral along the curve $A \to B \to C \to A$, which is the boundary of S.

Solution:
By Stokes theorem, the line integral can be computed as the flux of $\text{curl}(F) = \langle 1, 1, 1\rangle$. We compute $F(r(u, v)) \cdot r_u \times r_v = 3$ the result is 3 times the area of the triangle which is

Problem 14) (10 points)

What is the flux of the vector field

$$\vec{F}(x, y, z) = \langle 33x + \cos(y^2 \log(1 + y^2)), x^{9999}, \sqrt{2 + y^2 + \sin(\sin(xy^7))}\rangle$$

through the boundary S of the cone $E = \{x^2 + y^2 \leq z^2,\ 0 \leq z \leq 1\ \}$. All boundary parts of the cone are oriented so that the normal vector points outwards.

Solution:
We use the divergence theorem $\int\int_S F \cdot dS = \int\int\int \text{div}(F)\, dV$. The divergence is 3 so that the flux integral is 33 times the volume of the cone The cone has volume $\pi/3$. The flux is therefore $\boxed{33\pi/3 = 11\pi}$.