

| |
|-------|
| Name: |
|-------|

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Answers without derivation are not given credit.
- You have 180 minutes time to complete your work.

| | | |
|--------|--|-----|
| 1 | | 20 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| 10 | | 10 |
| 11 | | 10 |
| 12 | | 10 |
| 13 | | 10 |
| 14 | | 10 |
| 15 | | 10 |
| Total: | | 160 |

Problem 1) (20 points)

- 1) T F If $\vec{u} + \vec{v} + \vec{w} = \vec{0}$ then $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$.

Solution:

If $\vec{u} + \vec{v} + \vec{w} = \vec{0}$, then \vec{u} and \vec{v} and \vec{w} are in the same plane and the triple scalar product is zero, because it is the volume of the parallel epiped spanned by the three vectors.

- 2) T F $\int_0^5 \int_0^\pi r \, d\theta \, dr$ is half the area of a disc radius 5 in the plane.

Solution:

This is indeed the formula of the area in polar coordinates.

- 3) T F If a vector field $\vec{F}(x, y)$ satisfies $\text{curl}(\vec{F})(x, y) = 0$ for all points (x, y) in the plane, then \vec{F} is conservative.

Solution:

True. We have derived this from Greens theorem.

- 4) T F If the acceleration of a parameterized curve $\vec{r}(t) = (x(t), y(t), z(t))$ is zero then the curve $\vec{r}(t)$ is a line.

Solution:

If we integrate, then $\vec{r}'(t)$ is a constant vector \vec{v} . Then $\vec{r}(t) = \vec{r}(0) + t\vec{v}$.

- 5) T F A circle of radius 1/2 has a smaller curvature than a circle of radius 1.

Solution:

The curvature of a circle of radius r is equal to $1/r$.

- 6) T F The curve $\vec{r}(t) = (-\sin(t), \cos(t))$ for $t \in [0, \pi]$ is half a circle.

Solution:

True. Indeed, one can check that $\sin^2(t) + \cos^2(t) = 1$.

- 7) T F The function $u(t, x) = \sin(x + t)$ is a solution of the partial differential equation $u_{tx} + u = 0$

Solution:

$u_x = -\cos(x + t)$, $u_t = -\cos(x + t)$ and $u_{xt} + u = 0$.

- 8) T F The length of a curve $\vec{r}(t)$ in space parameterized on $a \leq t \leq b$ is the value of the integral $\int_a^b |\vec{T}'(t)| dt$, where $\vec{T}(t)$ is the unit tangent vector.

Solution:

The correct solution is $\int_a^b |\vec{r}'(t)| dt$.

- 9) T F Let (x_0, y_0) be the maximum of $f(x, y)$ under the constraint $g(x, y) = 1$. Then the gradient of g at (x_0, y_0) is parallel to the gradient of f at (x_0, y_0) .

Solution:

This paraphrases indeed part of the Lagrange equations.

- 10) T F At a point which is not a critical point, the directional derivative $D_{\vec{v}}f(x_0, y_0, z_0)$ can take both the negative and the positive sign.

Solution:

For $\vec{v} = \nabla f$, the directional derivative is positive. For $\vec{v} = -\nabla f$, the directional derivative is negative.

- 11) T F If a nonzero vector field $\vec{F}(x, y)$ is a gradient field, we always can find a curve C for which the line integral $\int_C \vec{F} \cdot d\vec{r}$ is positive.

Solution:

While there does not exist a closed curve with this property, there are many curves, if the field \vec{F} is not zero. Just move a bit into a direction of \vec{F} at a point, where \vec{F} is not zero.

- 12) T F If C is a closed level curve of a function $f(x, y)$ and $\vec{F} = (f_x, f_y)$ is the gradient field of f , then $\int_C \vec{F} \cdot d\vec{r} = 0$.

Solution:

The gradient field is perpendicular to the level curves.

- 13) T F The divergence of a gradient vector field $\vec{F}(x, y, z) = \nabla f(x, y, z)$ is always zero.

Solution:

Just take a simple example like $f(x, y, z) = x^2$, where $\text{div}(\text{grad}(f)) = 2$. Actually, $\text{div}(\text{grad}(f)) = \Delta f$ is the Laplacian of f .

- 14) T F The line integral of the vector field $\vec{F}(x, y, z) = \langle x^2, y^2, z^2 \rangle$ along a line segment from $(0, 0, 0)$ to $(1, 1, 1)$ is 1.

Solution:

By the fundamental theorem of line integrals, we can take the difference of the potential $f(x, y, z) = x^3/3 + y^3/3 + z^3/3$, which is $1/3 + 1/3 + 1/3$.

- 15) T F If $\vec{F}(x, y) = (x^2 - y, x)$ and $C : \vec{r}(t) = \langle \sqrt{\cos(t)}, \sqrt{\sin(t)} \rangle$ parameterizes the boundary of the region $R : x^4 + y^4 \leq 1$, then $\int_C \vec{F} \cdot d\vec{s}$ is twice the area of R .

Solution:

This is a direct consequence of Green's theorem and the fact that the two-dimensional curl $Q_x - P_y$ of $\vec{F} = \langle P, Q \rangle$ is equal to 2.

- 16) T F The flux of the vector field $\vec{F}(x, y, z) = \langle 0, y, 0 \rangle$ through the boundary S of a solid sphere E is equal to the volume the sphere.

Solution:

It is the **volume** of the solid torus.

- 17) T F The quadratic surface $-x^2 + y^2 + z^2 = 5$ is a one-sheeted hyperboloid.

Solution:

The traces are a circle and hyperboloids.

- 18) T F If \vec{F} is a vector field in space and S is the boundary of a sphere then the flux of $\text{curl}(\vec{F})$ through S is 0.

Solution:

This is true by Stokes theorem.

- 19) T F If $\text{div}(\vec{F})(x, y, z) = 0$ for all (x, y, z) and S is a torus surface, then the flux of \vec{F} through S is zero.

Solution:

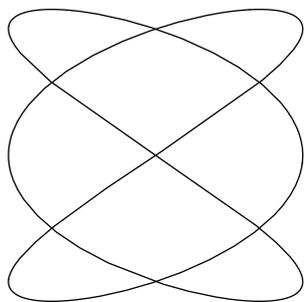
This is a consequence of the divergence theorem.

- 20) T F In spherical coordinates, the equation $\rho \cos(\phi) = \rho \cos(\theta) \sin(\phi)$ defines a plane.

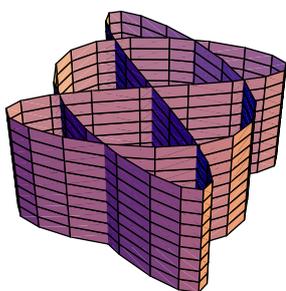
Solution:

True. It is the plane $z = x$.

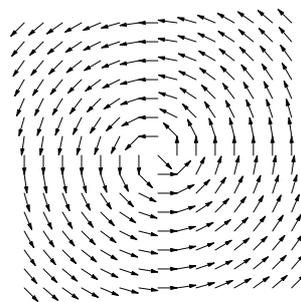
Problem 2) (10 points)



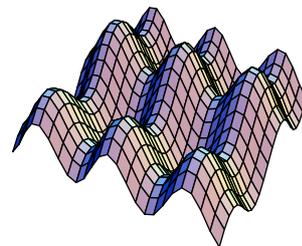
I



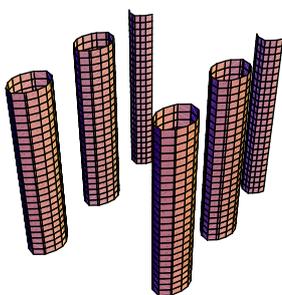
II



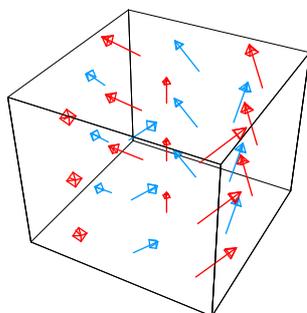
III



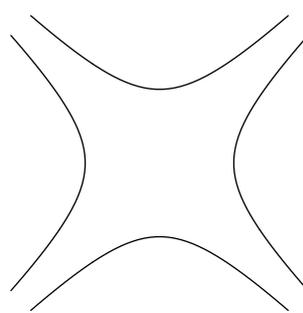
IV



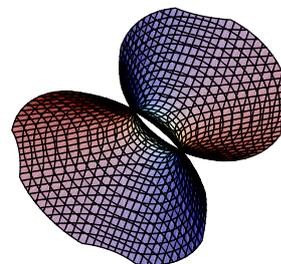
V



VI



VII



VIII

| | |
|--------------------------------------|---|
| Enter I,II,III,IV,V,VI,VII,VIII here | Equation |
| | $x^2 - y^2 + z^2 = 1$ |
| | $\vec{r}(t) = \langle \cos(3t), \sin(2t) \rangle$ |
| | $z = f(x, y) = \cos(3x) + \sin(2y)$ |
| | $\vec{F}(x, y) = \langle -y/\sqrt{x^2 + y^2}, x/\sqrt{x^2 + y^2} \rangle$ |
| | $\cos(3x) + \sin(2y) = 1$ |
| | $\vec{F}(x, y, z) = \langle -y, x, 1 \rangle$ |
| | $\vec{r}(u, v) = \langle \cos(3u), \sin(2u), v \rangle$ |
| | $\{(x, y) \in \mathbf{R}^2 \mid x^2 - y^2 = 1\}$ |

Solution:

| | |
|--------------------------------------|---|
| Enter I,II,III,IV,V,VI,VII,VIII here | Equation |
| VIII | $x^2 - y^2 + z^2 = 1$ |
| I | $\vec{r}(t) = \langle \cos(3t), \sin(2t) \rangle$ |
| IV | $z = f(x, y) = \cos(3x) + \sin(2y)$ |
| III | $\vec{F}(x, y) = \langle -y/\sqrt{x^2 + y^2}, x/\sqrt{x^2 + y^2} \rangle$ |
| V | $\cos(3x) + \sin(2y) = 1$ |
| VI | $\vec{F}(x, y, z) = \langle -y, x, 1 \rangle$ |
| II | $\vec{r}(u, v) = \langle \cos(3u), \sin(2u), v \rangle$ |
| VII | $\{(x, y) \in \mathbf{R}^2 \mid x^2 - y^2 = 1\}$ |

Furthermore, fill in the peoples names, Green, Stokes, Gauss, Fubini, Clairot. If there is no name associated to the theorem, write the name of the theorem.

| Formula | Name of the theorem |
|---|---------------------|
| $\int_C \vec{F} \cdot d\vec{r} = \int \int_S \text{curl}(\vec{F}) \cdot dS$ | |
| $f_{xy}(x, y) = f_{yx}(x, y)$ | |
| $\int_C \vec{F} \cdot d\vec{r} = \int \int_R \text{curl}(\vec{F}) \, dx dy$ | |
| $\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = f(\vec{r}(b)) - f(\vec{r}(a))$ | |
| $\int \int_S F \cdot dS = \int \int \int_E \text{div}(F) \, dV$ | |
| $\int_a^b \int_c^d f(x, y) \, dx dy = \int_c^d \int_a^b f(x, y) \, dy dx$ | |

Solution:

| | |
|---|---------------------------------------|
| Formula | Name of the theorem |
| $\int_C \vec{F} \cdot d\vec{r} = \int \int_S \text{curl}(\vec{F}) \cdot dS$ | Stokes |
| $f_{xy}(x, y) = f_{yx}(x, y)$ | Clairot |
| $\int_C \vec{F} \cdot d\vec{r} = \int \int_R \text{curl}(\vec{F}) \, dx dy$ | Green |
| $\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = f(\vec{r}(b)) - f(\vec{r}(a))$ | Fundamental theorem of line integrals |
| $\int \int_S F \cdot dS = \int \int \int_E \text{div}(F) \, dV$ | Gauss |
| $\int_a^b \int_c^d f(x, y) \, dx dy = \int_c^d \int_a^b f(x, y) \, dy dx$ | Fubini |

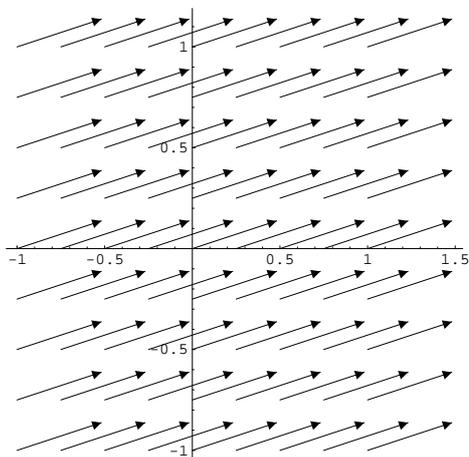
Problem 3) (10 points)

In this problem, vector fields \vec{F} are written as $\vec{F} = \langle P, Q \rangle$. We use abbreviations $\text{curl}(F) = Q_x - P_y$. When stating $\text{curl}(F) = 0$, we mean that $\text{curl}(F)(x, y) = 0$ vanishes for **all** (x, y) . Similarly, we say $\text{div}(F)$ if $\text{div}(F)(x, y) = P_x(x, y) + Q_y(x, y) = 0$ for all x, y .

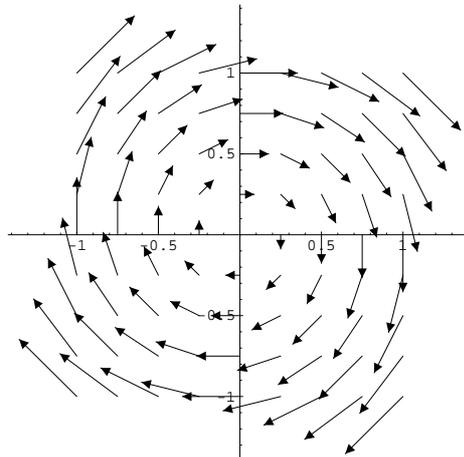
Check the box which match the formulas of the vector fields with the corresponding picture I,II,III or IV and mark also the places, indicating the vanishing of $\text{curl}(F)$.

| Vectorfield | I | II | III | IV | $\text{curl}(F) = 0$ | $\text{div}(F) = 0$ |
|---|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| $\vec{F}(x, y) = \langle 1, x \rangle$ | <input type="checkbox"/> |
| $\vec{F}(x, y) = \langle 3y, -3x \rangle$ | <input type="checkbox"/> |
| $\vec{F}(x, y) = \langle 7, 2 \rangle$ | <input type="checkbox"/> |
| $\vec{F}(x, y) = \langle x, y \rangle$ | <input type="checkbox"/> |

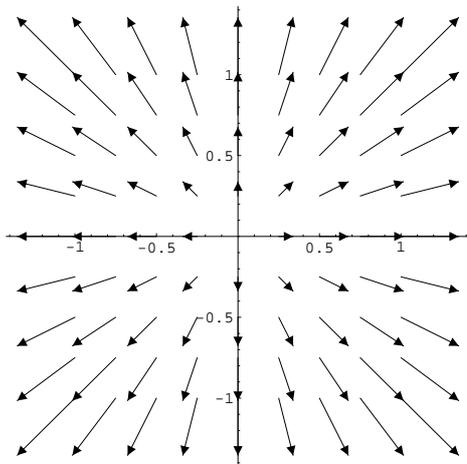
I



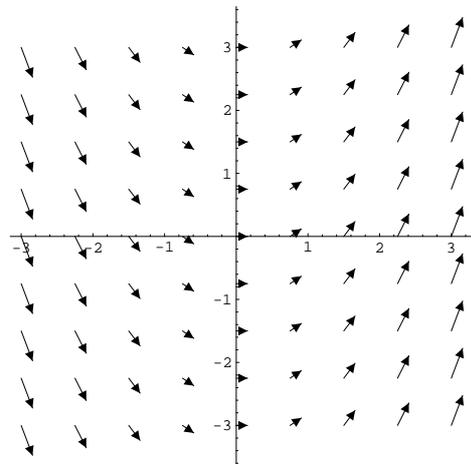
II



III



IV



Solution:

| Vectorfield | I | II | III | IV | $\text{curl}(F) = 0$ | $\text{div}(F) = 0$ |
|---|---|----|-----|----|----------------------|---------------------|
| $\mathbf{F}(x, y) = \langle y, 0 \rangle$ | | | | X | | X |
| $\mathbf{F}(x, y) = \langle y, -x \rangle$ | | X | | | | X |
| $\mathbf{F}(x, y) = \langle 0, 5 \rangle$ | X | | | | X | X |
| $\mathbf{F}(x, y) = \langle -x, -y \rangle$ | | | X | | X | |

Problem 4) (10 points)

a) (5 points) What is the area of the triangle A, B, P , where $A = (1, 1, 1), B = (1, 2, 3)$ and $P = (3, 2, 4)$?

b) (5 points) Find the distance between the point the point P and the line L passing through the points A with B .

Solution:

a) The area is half of the cross product of \vec{AB} and \vec{AP} which is $(0, 1, 2) \times (2, 1, 3)$ which is $|(1, 4, -2)|$ which is $\sqrt{21}$. The triangle has the area $\sqrt{21}/2$.

b) The distance formula is $|\vec{AB} \times \vec{AP}|/|\vec{AB}| = |(1, 4, -2)|/|(2, 1, 3)| = \frac{\sqrt{21}}{5}$.

Problem 5) (10 points)

The height of the ground near the Simplon pass in Switzerland is given by the function

$$f(x, y) = -x - \frac{y^3}{3} - \frac{y^2}{2} + \frac{x^2}{2}.$$

There is a lake in that area as you can see in the photo.

a) (7 points) Find and classify all the critical points of f and tell from each of them, whether it is a local maximum, a local minimum or a saddle point.

b) (3 points) For any pair of two different critical points A, B found in a) let $C_{a,b}$ be the line segment connecting the points, evaluate the line integral $\int_{C_{a,b}} \nabla f \cdot \vec{dr}$.



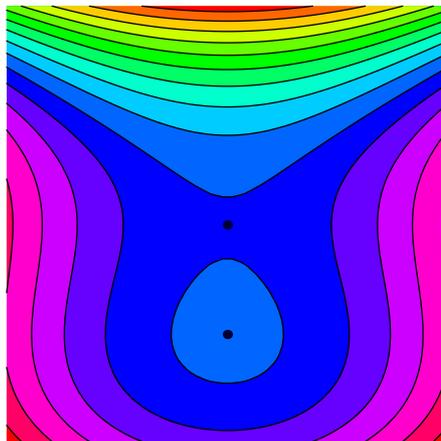
Photo of the lake in the Swiss alps near the Simplon mountain pass.

Solution:

a) The gradient is $\nabla f(x, y) = (x - 1, -y - y^2)$. This gradient vanishes if $x = 1$ and $y = -1$ or $y = 0$. So, there are two critical points $(1, -1), (1, 0)$. The Hessian matrix is

$$H(x, y) = \begin{bmatrix} 1 & 0 \\ 0 & -y - y^2 \end{bmatrix}.$$

| point | discriminant | f_{xx} | nature |
|-----------|--------------|----------|--------|
| $(1, -1)$ | D= 1 | 1 | min |
| $(1, 0)$ | D= -1 | 1 | saddle |



b) By the fundamental theorem of line integrals, the line integral between the two points is the difference of the potentials which is $f(1, -1) - f(1, 0) = (-1 + 1/3 - 1/2 + 1/2) - (-1 + 1/2) = -1/6$. Also the answer $1/6$ is correct of course, since we did not specify the direction.

Problem 6) (10 points)

a) (4 points) Find the linearization $L(x, y, z)$ of $f(x, y, z) = 2 + z - \sin(-x - 3y)$ at the point $P = (0, \pi, 2)$.

b) (4 points) Find the equation of the tangent plane at that point $P = (0, \pi, 2)$.

c) (2 points) Estimate $f(0.001, \pi, 2.02)$ using the linearization.

Solution:

a) $\nabla f(x, y, z) = (\cos(-x + 3y), 3 \cos(-x + 3y), 1)$. At the point $(0, \pi, 2)$, we have $\nabla f(0, \pi, 2) = (-1, -3, 1)$. We have $f(0, \pi, 2) = 4$. The linearization is

$$L(x, y, z) = 4 + (-1, -3, 1) \cdot (x, y - \pi, z - 2) = 2 + 3\pi - x - 3y + z .$$

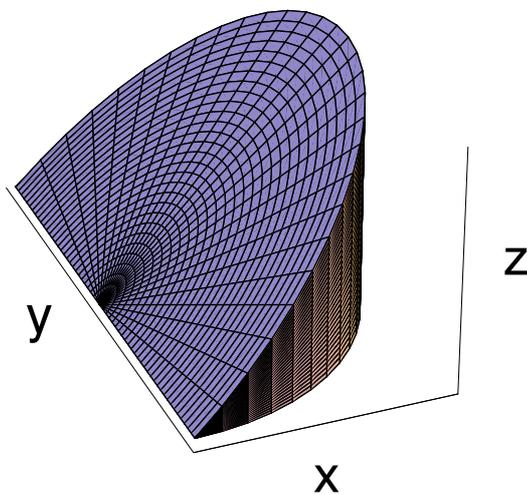
b) From the gradient $\nabla f(0, \pi, 2) = (a, b, d) = (-1, -3, 1)$, we know that the equation is $ax + by + cz = d$ which is $-x - 3y + z = d$. We get the constant d by plugging in the point. The equation is

$$-x - 3y + z = -3\pi + 2 .$$

c) Evaluate L at the point: $L(0.001, \pi, 2.02) = 4 + (-1, -3, 1) \cdot (0.001, 0, 0.02) = 4.019$.

| |
|------------------------|
| Problem 7) (10 points) |
|------------------------|

Find the volume of the wedge shaped solid that lies above the xy -plane and below the plane $z = x$ and within the solid cylinder $x^2 + y^2 \leq 9$.



Solution:

Use cylindrical coordinates and note that the wedge is positive on the right half plane so that we have to integrate over right half of the unit disc:

$$\int_0^3 \int_{-\pi/2}^{\pi/2} r^2 \cos(\theta) \, d\theta dr = 54/3 = 18 .$$

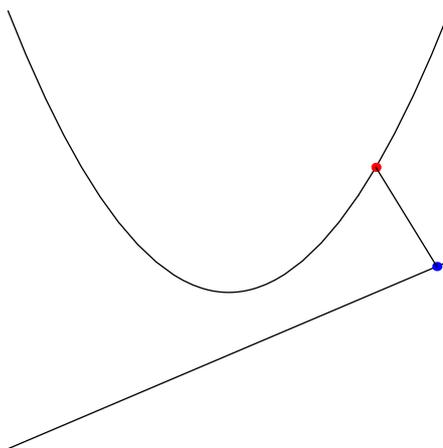
Note that spherical coordinates would not be good, since the bounds for the angle ϕ depend on the angle θ . The solution is 18.

Problem 8) (10 points)

The distance from a point (x, y) to the line $y = x$ in the plane is given by $f(x, y) = (y - x)/\sqrt{2}$. Use the Lagrange method to find the point (x, y) on the parabola

$$g(x, y) = x^2 - y = -2$$

which is closest to the line.

**Solution:**

We have to extremize the function $f(x, y)$ under the constraint $g(x, y) = -2$. The Lagrange equations are

$$-1/\sqrt{2} = \lambda 2x \tag{1}$$

$$1/\sqrt{2} = -\lambda \tag{2}$$

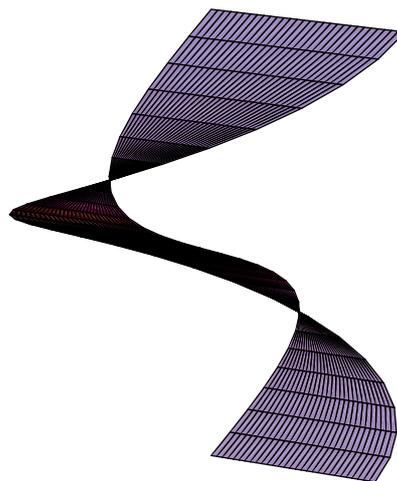
$$x^2 - y = -2 \tag{3}$$

We have $x = 1/2$ and $y = 9/4$. The point $(1/2, 9/4)$ on the parabola is closest to the line.

Problem 9) (10 points)

a) (5 points) A ribbon of a girl is modeled as a surface S which is parameterized by $\vec{r}(t, s) = (s \cos(t), \sin(t), t)$, where $t \in [0, 2\pi]$ and $s \in [0, 1]$. Find the surface area of this ribbon S .

b) (5 points) Part of the boundary of the ribbon is obtained when fixing $s = 1$. It is a curve in space. Find the arc length of this curve $\vec{r}(t)$, parametrized from $t = 0$ to 2π .



Painting: "Young Girl with Blue Ribbon" by the French painter Jean-Baptiste Greuze (1725-1805)

Solution:

a) We have to compute the integral $\int_0^{2\pi} \int_0^1 |r_s \times r_t| ds dt$. We have $r_s = (\cos(t), 0, 0)$ and $r_t = (-s \sin(t), \cos(t), 1)$. Now $r_s \times r_t = \cos(t)(0, -1, \cos(t))$ and $|r_s \times r_t| = |\cos(t)|\sqrt{1 + \cos^2(t)}$. The integral

$$2 \int_{-\pi/2}^{\pi/2} \cos(t) \sqrt{1 + \cos^2(t)} dt$$

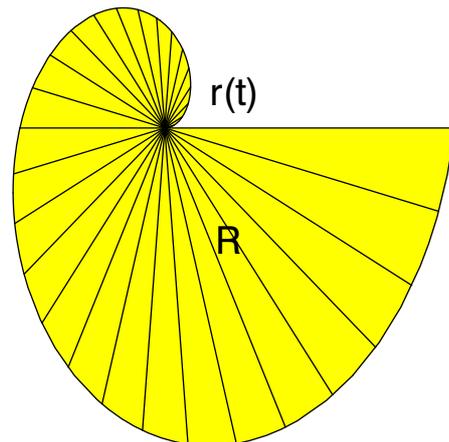
can be solved using integration by parts and some trigonometric identities to get $2 + \pi$. (We give full credit for the correct integral already).

b) For $s = 1$, we have a helix $\vec{r}(t) = (-\sin(t), \cos(t), t)$ which has the speed $|r'(t)| = \sqrt{2}$ and the arc length is $\int_0^{2\pi} \sqrt{2} dt = 2\pi\sqrt{2}$.

Problem 10) (10 points)

A region R in the xy -plane is given in polar coordinates by $0 \leq r(\theta) \leq \theta$ for $\theta \in [0, 2\pi]$. You see the region in the picture to the right. Its boundary is called the **Archimedes spiral**. It can be found on the tomb of Jacob Bernoulli. Evaluate the double integral

$$\iint_R \frac{e^{-x^2-y^2}}{(2\pi - \sqrt{x^2 + y^2})} dx dy .$$



Solution:

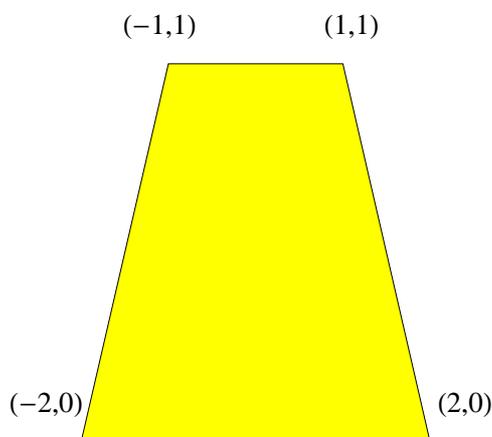
The region becomes a triangle in polar coordinates. Setting up the integral with $dA = drd\theta$ does not work. The integral $\int_0^{2\pi} \int_0^\theta e^{-r^2} drd\theta$ can not be solved. We have to change the order of integration:

$$\int_0^{2\pi} \int_r^{2\pi} e^{-r^2} r / (2\pi - r) d\theta dr$$

Evaluating the inner integral gives $\int_0^{2\pi} e^{-r^2} r dr = \boxed{(1 - e^{-4\pi^2})2}$.

Problem 11) (10 points)

Find the line integral of the vector field $\vec{F}(x, y) = \langle 3y, 8x \rangle$ along the boundary of the trapezoid with vertices $(-2, 0), (2, 0), (1, 1), (-1, 1)$.



Solution:

The curl is constant 5 so that by Green's theorem, the line integral is the area of the region which is $5 \cdot 3 = 15$.

Problem 12) (10 points)

Let \vec{F} be the vector field $\vec{F}(x, y, z) = \langle -z + x^{(x^x)}, 5 + y^{(y^y)}, y + z^{(z^z)} \rangle$.

Let C be the curve given by the parameterization $\vec{r}(t) = \langle \cos(t), 0, \sin(t) \rangle$, for $0 \leq t \leq 2\pi$.

Compute the line integral of \vec{F} along C .

Hint. You might want to consider a surface contained in the xz -plane which is enclosed by the curve.

Solution:

We use Stokes Theorem. The curl of the vector field is $\langle 1, -1, 0 \rangle$. The parameterization describes the circle $x^2 + z^2 = 1$, where $y = 0$. The curve starts at $(1, 0, 0)$ and rotates towards back towards $(0, 0, 1)$. By Stokes theorem, the line integral can be computed as the flux of $\text{curl}(\vec{F})$ through the unit disk D in the xz plane which has the normal vector $\vec{r}_u \times \vec{r}_v = -\vec{j}$ and $\text{curl}(\vec{F})(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) = -1$. The flux is

$$\iint_D -1 \, dx \, dz = -\pi .$$

The final answer is $\boxed{-\pi}$.

Problem 13) (10 points)

What is the flux of the vector field

$$\vec{F}(x, y, z) = \langle 3x + \cos(z^2 \sin(z)), x, \sin(y^3 + \cos(\sin(xy^3))) \rangle$$

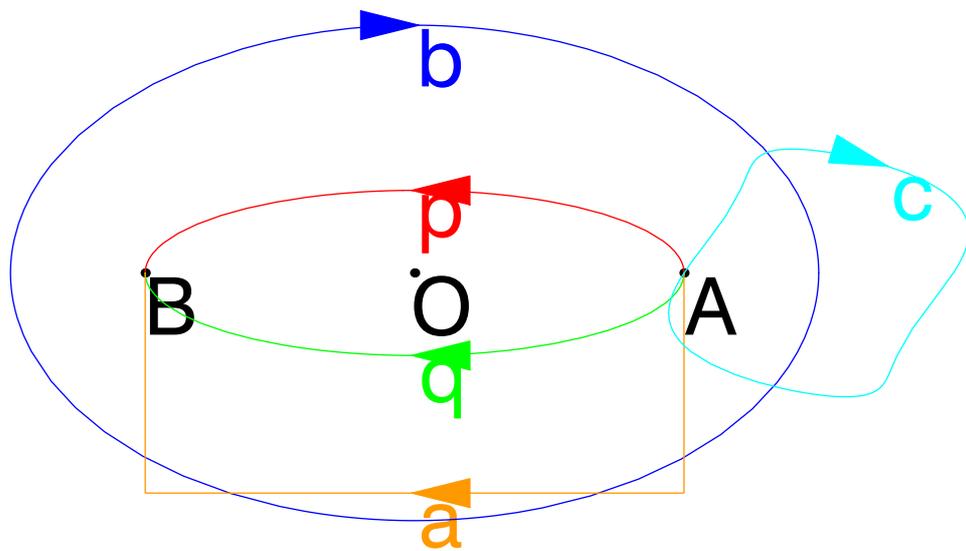
through the boundary S of the solid cylinder $E = \{x^2 + y^2 \leq 1, 0 \leq z \leq 10\}$. The surface of the cylinder is oriented so that the normal vector points outwards.

Solution:

We use the divergence theorem $\iint_S \vec{F} \cdot d\vec{S} = \iiint_V \text{div}(\vec{F}) \, dV$. The divergence is 3 so that the flux integral is 3 times the volume of the cylinder. The cylinder has volume $10 \cdot \pi$. The flux is therefore $\boxed{30\pi}$.

Problem 14) (10 points)

Suppose \vec{F} is an irrotational vector field in the plane (that is, its curl is everywhere zero) that is not defined at the origin $O = (0, 0)$. Suppose the line integral of \vec{F} along the path p from A to B is 5 and the line integral of \vec{F} along the path q from A to B is -4 . Find the line integral of \vec{F} along the following three paths:



- a) (3 points) The path a from A to B going clockwise below the origin.
- b) (4 points) The closed path b encircling the origin in a clockwise direction.
- c) (3 points) The closed path c starting at A and ending in A without encircling the origin.

Solution:

- a) The result is the same for the path a and the path q . The vector field is conservative in the lower half plane. The result is $\boxed{-4}$. b) The line integral is the same as the difference of the line integral along q and the line integral along p which is $-4 - 5 = \boxed{-9}$. The path $q - p$ encircles the origin in the same direction than the path b . Because the curl is 0 in the region enclosed by these two curves, Greens theorem assures that the line integrals are the same.
- c) The vector field F is conservative in the righth half plane. By the fundamental theorem of line integrals or using the closed loop property, the result is $\boxed{0}$.

| |
|-------------------------|
| Problem 15) (10 points) |
|-------------------------|

Let S be the graph of the function $f(x, y) = 2 - x^2 - y^2$ which lies above the disk $\{(x, y) \mid x^2 + y^2 \leq 1\}$ in the xy -plane. The surface S is oriented so that the normal vector points upwards. Compute the flux $\int \int_S \vec{F} \cdot d\vec{S}$ of the vectorfield

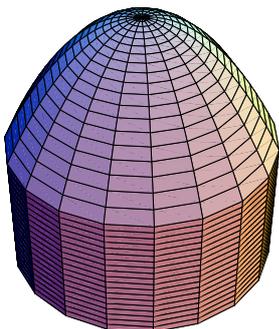
$$\vec{F} = \left(-4x + \frac{x^2 + y^2 - 1}{1 + 3y^2}, 3y, 7 - z - \frac{2xz}{1 + 3y^2} \right)$$

through S using the divergence theorem.

Solution:

We apply the divergence theorem to the region $E = \{0 \leq z \leq f(x, y), x^2 + y^2 \leq 1\}$. Using $\text{div}(F) = -2$, we get

$$\begin{aligned} \iiint \text{div}(F) dV &= \int_0^1 \int_0^{2\pi} \int_0^{2-r^2} (-2) r dz d\theta dr \\ &= (-2) \int_0^1 \int_0^{2\pi} (2-r^2) r dz d\theta dr \\ &= (-2)(2\pi)(2/2 - 1/4) = -3\pi. \end{aligned}$$



By the divergence theorem, this is the flux of

F through the boundary of E which consists of the surface S , the cylinder $S_1 : r(u, v) = (\cos(u), \sin(u), v)$ with normal vector $r_u \times r_v = (-\sin(u), \cos(u), 0) \times (0, 0, 1) = (\cos(u), \sin(u), 0)$ plus the flux through the floor $S_2 : \vec{r}(u, v) = (v \sin(u), v \cos(u), 0)$ with normal vector $r_u \times r_v = (0, 0, -v)$. The flux through S_1 is

$$\begin{aligned} \iint_{S_1} \vec{F} \cdot dS &= \int_0^1 \int_0^{2\pi} F(\cos(u), \sin(u), v) \cdot (\cos(u), \sin(u), 0) dudv \\ &= \int_0^1 \int_0^{2\pi} (-4 \cos^2(u) + 3 \sin^2(u)) dudv = -\pi \end{aligned}$$

The flux through S_2 is

$$\begin{aligned} \iint_{S_2} \vec{F} \cdot dS &= \int_0^1 \int_0^{2\pi} F(v \sin(u), v \cos(u), 7) \cdot (0, 0, -v) dudv \\ &= \int_0^1 \int_0^{2\pi} (-7v) dudv = -7\pi \end{aligned}$$

By the divergence theorem, $\iint_S \vec{F} \cdot dS + \iint_{S_1} \vec{F} \cdot dS + \iint_{S_2} \vec{F} \cdot dS = -3\pi$ so that $\iint_S \vec{F} \cdot dS = -3\pi + \pi + 7\pi = \boxed{5\pi}$.