8/6/2009 FINAL EXAM **Maths 21a, O. Knill, Summer 2009**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

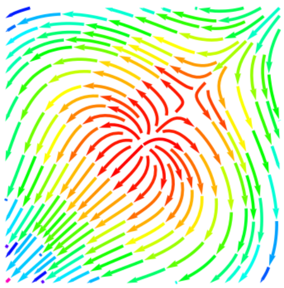
1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points)

1)	T	F	The line $\vec{r}(t) = \langle t, t, t \rangle$ is perpendicular to the plane $x + y + z = 10$.
2)	T	F	The quadratic surface $-x^2 + y^2 + z^2 = -1$ is a one sheeted hyperboloid.
3)	T	F	The relation $\lvert\vec{u} \times \vec{v}\rvert = \lvert\vec{u} \cdot \vec{v}\rvert$ is only possible if at least one of the vectors $\vec{u}$ and $\vec{v}$ is the zero vector.
4)	T	F	$\int_0^{\pi/2} \int_0^1 r^3 \, d\theta \, dr = \int_0^1 \int_0^1 x^2 + y^2 \, dxdy$.
5)	T	F	If a vector field $\vec{F}(x, y)$ satisfies $\text{curl}(\vec{F})(x, y) = Q_x(x, y) - P_y(x, y) = 0$ and $\text{div}(\vec{F})(x, y) = P_x(x, y) + Q_y(x, y) = 0$ for all points (x, y) in the plane, then $\vec{F}$ is a constant field.
6)	T	F	The acceleration vector $\vec{r}\,''(t) = \langle x(t), y(t) \rangle$, the velocity vector $\vec{r}\,'(t)$ and $\vec{r}\,'(t) \times \vec{r}\,''(t)$ form three vectors which are mutually perpendicular.
7)	T	F	The curvature of the curve $\vec{r}(t) = \langle \sin(2t), 0, \cos(2t) \rangle$ is equal to the curvature of the curve $\vec{s}(t) = \langle 0, \cos(3t), \sin(3t) \rangle$.
8)	T	F	The space curve $\vec{r}(t) = \langle t\sin(t), t\cos(t), t^2 \rangle$ for $t \in [0, 10\pi]$ is located on a cone.
9)	T	F	If a smooth function $f(x, y)$ has a global maximum, then this maximum is a critical point.
10)	T	F	If $L(x, y)$ is the linearization of $f(x, y)$ and $\vec{s}(t)$ is the line tangent to the curve $\vec{r}(t)$ at t_0. Then $d/dt L(\vec{s}(t)) = d/dt f(\vec{r}(t))$ at the time $t = t_0$.
11)	T	F	If $\vec{F}$ is a gradient field and $\vec{r}(t)$ is a flow line defined by $\vec{r}\,'(t) = \vec{F}(\vec{r}(t))$, then the line integral $\int_0^1 \vec{F} \cdot d\vec{r}$ is either positive or zero.
12)	T	F	If we extremize the function $f(x, y)$ under the constraint $g(x, y) = 1$, and the functions are the same $f = g$, we have infinitely many extrema.
13)	T	F	If a point (x_0, y_0) is a critical point of $f(x, y)$ under the constraint $g(x, y) = 1$, then it is also a critical point of the function $f(x, y)$ without constraints.
14)	T	F	If a vector field $\vec{F}(x, y)$ is a gradient field, then any line integral along any ellipse is zero.
15)	T	F	The flux of an irrotational vector field is zero through any surface S in space.
16)	T	F	The divergence of a gradient field $\vec{F}(x, y) = \nabla f(x, y)$ is zero.
17)	T	F	The line integral of a vector field $\vec{F}(x, y, z) = \langle x, y, z \rangle$ along a circle in the $xy-$ plane is zero.
18)	T	F	For any solid E, the moment of inertia $\iiint_E x^2 + y^2 \, dxdydz$ is always larger than the volume $\iiint_E 1 \, dxdydz$.
19)	T	F	The curvature of a circle is always larger than the acceleration.
20)	T	F	The directional derivative of $\text{div}(\vec{F}(x, y))$ of the divergence of the vector field $\vec{F} = \langle P, Q \rangle$ in the direction $\vec{v} = \langle 1, 0 \rangle$ is $P_{xx} + Q_{xy}$.

Problem 2) (10 points) Match objects with definitions. No justifications necessary.

Match the objects with their definitions



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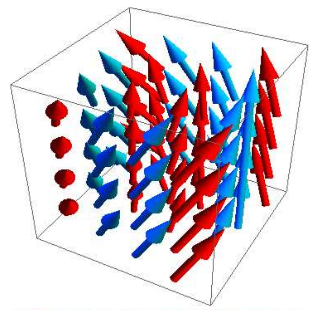

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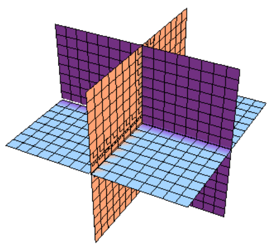

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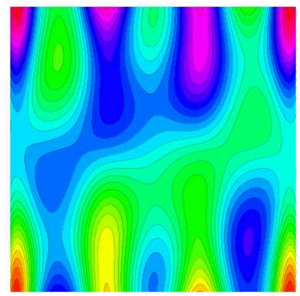

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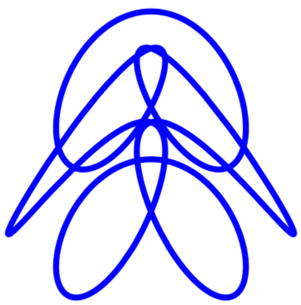

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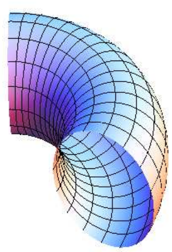

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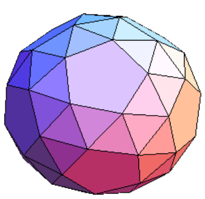

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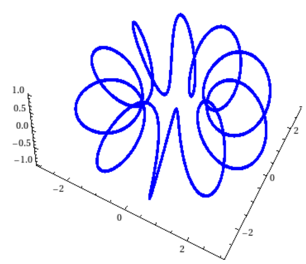


8

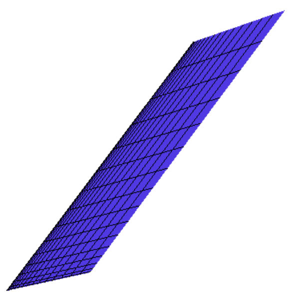
Enter 1-8 or 0 if no match	Object definition
	$\vec{r}(t) = \langle (2+\cos(10t))\cos(t), (2+\cos(10t))\sin(t), \sin(10t) \rangle$
	$\vec{F}(x,y,z) = \langle -y, x, 2 \rangle$
	$\vec{r}(t,s) = \langle (2+\cos(s))\cos(t), (2+\cos(s))\sin(t), \sin(s) \rangle$
	$\{(x,y,z) \mid \sin(x^2) - \cos(y^2) = 1 \}$
	$\vec{F}(x,y) = \langle x - y^2, y - x^2 \rangle$
	$xyz = 0$
	$x^2 + y^2 - z^2 = 1$
	$\{(x,y) \mid \sin(x^2\sin(x))y + \sin(y-x) = c \}$
	$\vec{r}(t) = \langle \sin(t) + \cos(5t), \cos(t) + \cos(6t) \rangle$

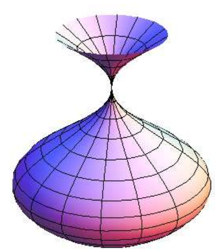
Problem 3) (10 points)

a) (5 points) Check every box to the left, for which the missing part to the right is $\nabla f(1,2)$. The function $f(x,y)$ is an arbitrary nice function like for example $f(x,y) = x - yx + y^2$. The curve $\vec{r}(t)$, wherever it appears, parametrizes the level curve $f(x,y) = f(1,2)$ and has the property that $\vec{r}(0) = \langle 1, 2 \rangle$.

Check	Topic	Statement
	Linearization	$L(x,y) = f(1,2) + \square \cdot \langle x-1, y-2 \rangle$
	Chain rule	$\frac{d}{dt} f(\vec{r}(t))\|_{t=0} = \square \cdot \vec{r}\,'(0)$
	Steepest descent	f decreases at $(1,2)$ most in the direction of $\square$
	Estimation	$f(1+0.1, 1.99) \sim f(1,2) + \square \cdot \langle 0.1, -0.01 \rangle$
	Directional derivative	$D_{\vec{v}} f(1,2) = \square \cdot \vec{v}$
	Level curve	of f through $(1,2)$ has the form $\square \cdot \langle x-1, y-2 \rangle = 0$
	Vector projection	of $\nabla f(1,2)$ onto $\vec{v}$ is $\vec{v}(\vec{v} \cdot \square)/\|\vec{v}\|^2$
	Tangent line	of $\vec{r}(t)$ at $(1,2)$ is parametrized by $\vec{R}(s) = \langle 1, 2 \rangle + s \, \square$

b) (5 points) The surfaces are given either as a parametrization or implicitly. Match them. Each surface matches one definition.


A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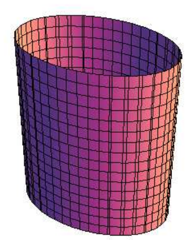
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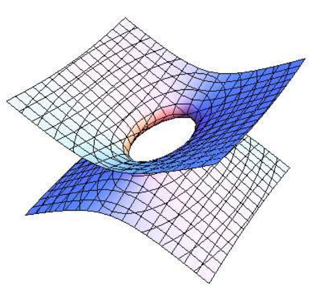
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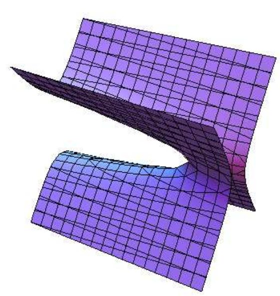
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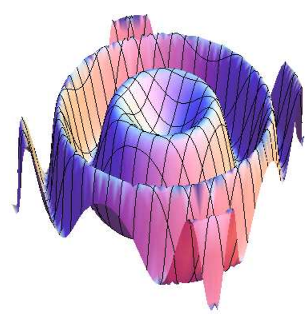
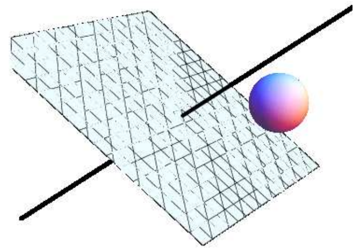

F

Enter A-F here	Function or parametrization
	$\vec{r}(u,v) = \langle u^2, v^2, u^2 + v^2 \rangle$
	$\vec{r}(u,v) = \langle (1+\sin(u))\cos(v), (1+\sin(u))\sin(v), u \rangle$
	$4x^2 + y^2 - 9z^2 = 1$
	$x - 9y^2 + 4z^2 = 1$
	$\vec{r}(u,v) = \langle u, v, \sin(u^2+v^2) \rangle$
	$4x^2 + 9y^2 = 1$

Problem 4) (10 points)

We want to determine whether the distance of the sphere S of radius 1 centered at $P = (1, 2, 3)$ to the plane $E : x + y + z = 1$ is larger than the distance of the same sphere to the line $L : x + y = y + z = x + z$.



a) (5 points) Find the distance from the sphere S to the plane E.

b) (5 points) Find the distance from the sphere S to the line L.

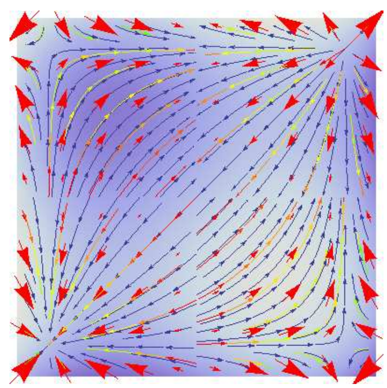
Problem 5) (10 points)

Where does the vector field

$$\vec{F}(x, y) = \langle P, Q \rangle = \langle y(x^3 - 3x), x(y^3 - 3y) \rangle$$

have maximal or minimal curl

$$f(x, y) = \text{curl}(\vec{F})(x, y) = Q_x(x, y) - P_y(x, y) .$$



a) (8 points) Find all extrema and determine whether they are maxima, minima or saddle points.

b) (2 points) Is there a global maximum of $f(x, y)$?

Problem 6) (10 points)

A sprinkler at position $(0, 0, 1)$ throws out water with constant speed and elevation angle 45 degrees. The water is under constant gravitational acceleration $\langle 0, 0, -10 \rangle$.

a) (5 points) Find the trajectory $\vec{r}(t)$, if the initial velocity is $\vec{r}\,'(t)|_{t=0} = \langle \cos(\theta), \sin(\theta), 1 \rangle$ and write down the formula for the arc length from $t = 0$ to $t = 1$. You do not have to start evaluating the integral.

b) (5 points) All the trajectories together form a surface $\vec{r}(\theta, t)$. Parametrize this surface and write down the formula for the surface area if $0 \leq t \leq 1$ and $0 \leq \theta \leq 2\pi$. You do not have to start evaluating the integral.

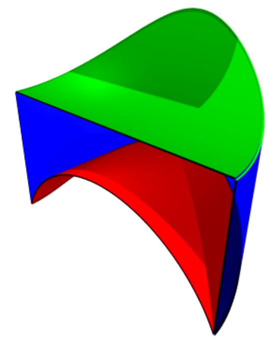
Problem 7) (10 points)

Compute the integral

$$\iiint_E x^2 \, dzdxdy ,$$

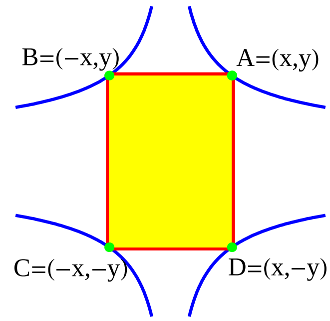
over the solid E defined by the four conditions

$$x^2 + y^2 \leq 1, y \geq 0, z < 4 - y^2, \text{ and } z > -5 + x^2 .$$



Problem 8) (10 points)

Use the method of Lagrange multipliers to find the centrally symmetric rectangle with corners $A = (x, y), B = (-x, y), C = (-x, -y), D = (x, -y)$ on the curve $g(x, y) = x^2y^4 = 1$ which has minimal circumference $f(x, y) = 4x + 4y$.



Problem 9) (10 points)

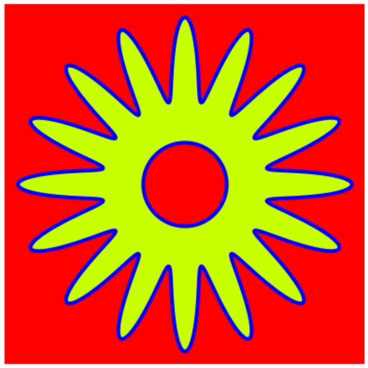
a) (5 points) Find the area of the region which is given in polar coordinates (r, θ) as

$$1 \leq r \leq 2 + \cos(16\theta) .$$

The picture of this region can be admired to the right.

b) (5 points) Find

$$\int_0^{\pi/2} \int_x^{\pi/2} \sin(y)/y \, dydx .$$

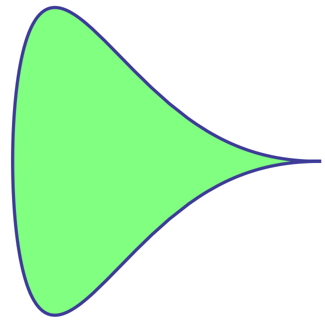


Problem 10) (10 points)

Find the area of the region enclosed by

$$\vec{r}(t) = \langle t^2, \frac{(\sin(\pi t))^2}{t} \rangle$$

for $-1 \leq t \leq 1$. Use an integral theorem with a suitable vector field.



Problem 11) (10 points)

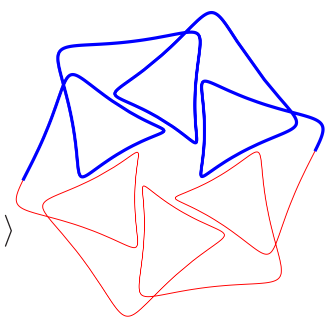
Compute the line integral of the vector field

$$\vec{F}(x, y) = \langle 3x + 2xy^2, 2y + 2x^2y \rangle$$

along the curve

$$\vec{r}(t) = \langle 6\cos(t)+4\cos(7t)+\sin(17t), 6\sin(t)+4\sin(7t)+\cos(17t) \rangle$$

from $t = 0$ to $t = \pi$.



Problem 12) (10 points)

Find the flux of vector field

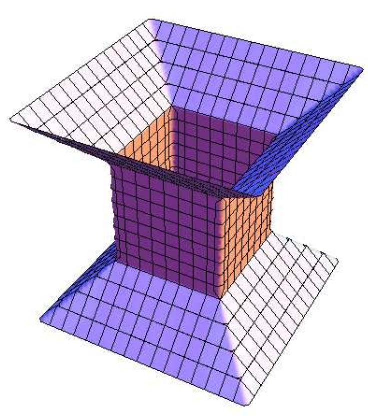
$$\vec{F}(x, y, z) = \langle y^2 \sin(z) - x, z^2 \cos(x^2), 5z \rangle$$

through the surface S given by the $p \to infty$ limit of

$$|x|^p + |y|^p - |z|^p = 1, \; -2 < z < 2 \; .$$

The surface is oriented that the normal vectors points outwards.

Hints. The surface (see picture) becomes closed if two not yet included square "lids" at $z = 2$ and $z = -2$ with corners at $(\pm 2, \pm 2, \pm 2)$ are added. You can use without proof that the volume of the solid is $(16 + 4)/2 + (16 + 4)/2 + 2 * 2 = 20 + 4 = 24$.



Problem 13) (10 points)

Compute the flux of the curl of the vector field

$$\vec{F}(x,y,z) = \langle x^2, y, \sin(z^2)^4 \rangle$$

through the surface which has the parametrization

$$\vec{r}(t,s) = \langle s\cos(t), s\sin(t), 3\sin(s)\cos(t) \rangle ,$$

where $0 \leq t \leq 2\pi$ and $0 \leq s \leq 2\pi$.