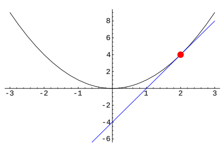
TANGENT LINES AND PLANES — Maths21a

TANGENT LINE. Because $\vec{n} = \nabla f(x_0, y_0) = \langle a, b \rangle$ is perpendicular to the level curve $f(x,y) = c$ through (x_0, y_0), the equation for the tangent line is

$$\boxed{ax + by = d,\ a = f_x(x_0, y_0),\ b = f_y(x_0, y_0),\ d = ax_0 + by_0}$$

EXAMPLE: Find the tangent to the graph of the function $g(x) = x^2$ at the point $(2,4)$. Solution: the level curve $f(x,y) = y - x^2 = 0$ is the graph of a function $g(x) = x^2$ and the tangent at a point $(2, g(2)) = (2,4)$ is obtained by computing the gradient $\langle a, b \rangle = \nabla f(2,4) = \langle -g'(2), 1 \rangle = \langle -4, 1 \rangle$ and forming $-4x + y = d$, where $d = -4 \cdot 2 + 1 \cdot 4 = -4$. The answer is $\boxed{-4x + y = -4}$ which is the line $y = 4x - 4$ of slope 4. Graphs of 1D functions are curves in the plane, you have computed tangents in single variable calculus.

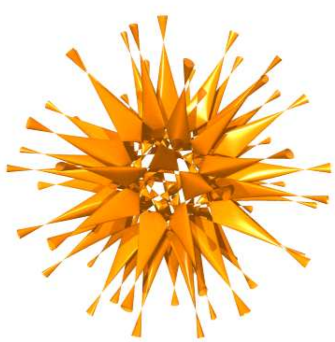


GRADIENT IN 3D. If $f(x,y,z)$ is a function of three variables, then $\nabla f(x,y,z) = (f_x(x,y,z), f_y(x,y,z), f_z(x,y,z))$ is called the **gradient** of f.

POTENTIAL AND FORCE. Force fields F in nature often are gradients of a function $U(x,y,z)$. The function U is called a **potential** of F or the potential energy.

EXAMPLE. If $U(x,y,z) = 1/|x|$, then $\nabla U(x,y,z) = -x/|x|^3$. The function $U(x,y,z)$ is the **Coulomb potential** and ∇U is the **Coulomb force**. The gravitational force has the same structure but a different constant. While much weaker, it is more effective because it only appears as an attractive force, while electric forces can be both attractive and repelling.

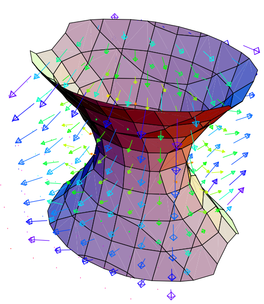
LEVEL SURFACES. If $f(x,y,z)$ is a function of three variables, then $f(x,y,z) = C$ is a surface called a level surface of f. The picture to the right shows the Barth surface $(3+5t)(-1+x^2+y^2+z^2)^2\ (-2+t+x^2+y^2+z^2)^2\ 8(x^2-t^4y^2)(-t^4x^2+z^2)\ (y^2-t^4z^2)\ (x^4-2x^2y^2+y^4-2x^2z^2-2y^2z^2+z^4) = 0$, where $t = (\sqrt{5}-1)/2$ is the golden ratio.



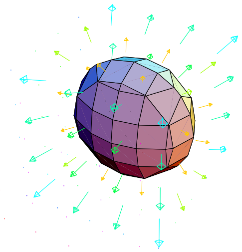
ORTHOGONALITY OF GRADIENT. We have seen that the gradient $\nabla f(x,y)$ is normal to the level curve $f(x,y) = c$. This is also true in 3 dimensions:

> The gradient $\nabla f(x,y,z)$ is normal to the level surface $f(x,y,z)$.

The argument is the same as in 2 dimensions: take a curve $\vec{r}(t)$ on the level surface. Then $\frac{d}{dt} f(\vec{r}(t)) = 0$. The chain rule tells from this that $\nabla f(x,y,z)$ is perpendicular to the velocity vector $\vec{r}'(t)$. Having ∇f tangent to all tangent velocity vectors on the surfaces forces it to be orthogonal.

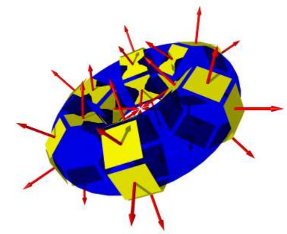


EXAMPLE. The gradient of $f(x,y,z) = x^2 + 2y^2 + z^2$ at a point (x,y,z) is $(2x, 4y, 2z)$. It illustrates well that going into the direction of the gradient **increases** the value of the function.



TANGENT PLANE. Because $\vec{n} = \nabla f(x_0, y_0, z_0) = \langle a, b, c \rangle$ is perpendicular to the level surface $f(x,y,z) = C$ through (x_0, y_0, z_0), the equation for the tangent plane is

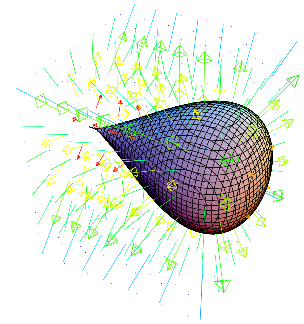
$$\boxed{ax + by + cz = d,\quad (a,b,c) = \nabla f(x_0, y_0, z_0),\ d = ax_0 + by_0 + cz_0.}$$



EXAMPLE. Find the general formula for the tangent plane at a point (x,y,z) of the Barth surface. Just kidding ... Note however that computing this would be no big deal with the help of a computer algebra system like Mathematica. Lets look instead at the quartic surface

$$f(x,y,z) = x^4 - x^3 + y^2 + z^2 = 0$$

which is also called the "piriform" or "pair shaped surface". What is the equation for the tangent plane at the point $P = (2,2,2)$? We get $\langle a, b, c \rangle = \langle 20, 4, 4 \rangle$ and so the equation of the plane $20x + 4y + 4z = 56$.



EXAMPLE. An important example of a level surface is $g(x,y,z) = z - f(x,y)$ which is the graph of a function of two variables. The gradient of g is

$$\nabla g = (-f_x, -f_y, 1)$$

By writing the graph as a level curve of a function of three variables, we could find the equation of the tangent plane at a point. This is one of the reasons, why we want to treat graphs as level surfaces too.