

## LINES and PLANES

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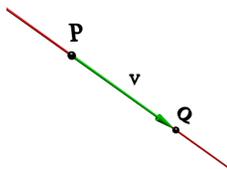
LINES. A point  $P$  and a vector  $\vec{v}$  define a line  $L$ . It is the set of points

$$L = \{P + t\vec{v}, \text{ where } t \text{ is a real number}\}$$

The line contains the point  $P$  and points into the direction  $\vec{v}$ .

EXAMPLE.  $L = \{(x, y, z) = (1, 1, 2) + t(2, 4, 6)\}$ .

This description is called the **parametric equation** for the line.



EQUATIONS OF LINE. We can write  $(x, y, z) = (1, 1, 2) + t(2, 4, 6)$  so that  $x = 1 + 2t, y = 1 + 4t, z = 2 + 6t$ . If we solve the first equation for  $t$  and plug it into the other equations, we get  $y = 1 + (2x - 1), z = 2 + 3(2x - 1)$ . We can therefore describe the line also as

$$L = \{(x, y, z) \mid y = 2x - 1, z = 6x - 4\}$$

SYMMETRIC EQUATION. The line  $\vec{r} = P + t\vec{v}$  with  $P = (x_0, y_0, z_0)$  and  $\vec{v} = (a, b, c)$  satisfies the **symmetric equations**  $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$  (every expression is equal to  $t$ ).

PROBLEM. Find the equations for the line through the points  $P = (0, 1, 1)$  and  $Q = (2, 3, 4)$ .

SOLUTION. The parametric equations are  $(x, y, z) = (0, 1, 1) + t(2, 2, 3)$  or  $x = 2t, y = 1 + 2t, z = 1 + 3t$ . Solving each equation for  $t$  gives the symmetric equations  $x/2 = (y - 1)/2 = (z - 1)/3$ .

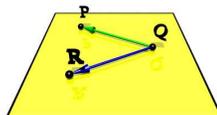
PLANES. A point  $P$  and two vectors  $\vec{v}, \vec{w}$  define a plane  $\Sigma$ . It is the set of points

$$\Sigma = \{P + t\vec{v} + s\vec{w}, \text{ where } t, s \text{ are real numbers}\}$$

The plane contains the point  $P$ .

EXAMPLE.  $\Sigma = \{(x, y, z) \mid (1, 1, 2) + t(2, 4, 6) + s(1, 0, -1)\}$ .

This is called the **parametric description** of a plane.



EQUATION OF PLANE. Given a plane as a parametric equation  $P = Q + t\vec{v} + s\vec{w}$ . The vector  $\vec{n} = \vec{v} \times \vec{w}$  is orthogonal to both  $\vec{v}$  and  $\vec{w}$ . Because also the vector  $PQ = Q - P$  is perpendicular to  $\vec{n}$  we have  $(Q - P) \cdot \vec{n} = 0$ . With  $Q = (x_0, y_0, z_0)$ ,  $P = (x, y, z)$ , and  $\vec{n} = (a, b, c)$ , this means  $ax + by + cz = ax_0 + by_0 + cz_0 = d$ . The plane is therefore described by a single equation  $ax + by + cz = d$ .

PROBLEM. Find the equation of a plane which contains the three points  $P = (-1, -1, 1), Q = (0, 1, 1), R = (1, 1, 3)$ .

SOLUTION. The plane contains the two vectors  $\vec{v} = (1, 2, 0)$  and  $\vec{w} = (2, 2, 2)$ . We have  $\vec{n} = (4, -2, -2)$  and the equation is  $4x - 2y - 2z = d$ . The constant  $d$  is obtained by plugging in one point:  $4x - 2y - 2z = -4$ .

### LINES AND PLANES IN MATHEMATICA

Plotting a line: `ParametricPlot3D[{1, 1, 1} + t{3, 4, 5}, {t, -2, 2}]`

Plotting a plane: `ParametricPlot3D[{1, 1, 1} + t{3, 4, 5} + s{1, 2, 3}, {t, -2, 2}, {s, -2, 2}]`

Finding the equation of a plane

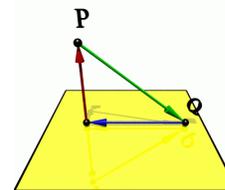
`P = {-1, -1, 1}; Q = {0, 1, 1}; R = {1, 1, 3}; n = Cross[Q - P, R - P]; n.{x, y, z} - n.P`

DISTANCE POINT-PLANE (3D). If  $P$  is a point in space and  $\vec{n} \cdot \vec{x} = d$  is a plane containing a point  $Q$ , then

$$d(P, L) = |(P - Q) \cdot \vec{n}| / |\vec{n}|$$

is the distance between  $P$  and the plane.

You recognize that this is just the scalar projection of the vector  $\vec{QP} = P - Q$  onto the vector  $\vec{n}$ .

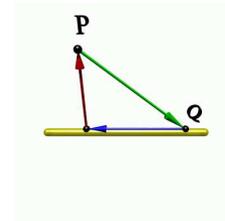


DISTANCE POINT-LINE (3D). If  $P$  is a point in space and  $L$  is the line  $\vec{r}(t) = Q + t\vec{u}$ , then

$$d(P, L) = |(P - Q) \times \vec{u}| / |\vec{u}|$$

is the distance between  $P$  and the line  $L$ .

This formula is verified by writing  $(P - Q) \times \vec{u} = |P - Q| |\vec{u}| \sin(\theta)$ .

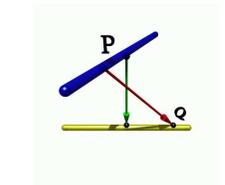


DISTANCE LINE-LINE (3D).  $L$  is the line  $\vec{r}(t) = Q + t\vec{u}$  and  $M$  is the line  $\vec{s}(t) = P + t\vec{v}$ , then

$$d(L, M) = |(P - Q) \cdot (\vec{u} \times \vec{v})| / |\vec{u} \times \vec{v}|$$

is the distance between the two lines  $L$  and  $M$ .

This formula is just the scalar projection of  $\vec{QP} = P - Q$  onto the vector  $\vec{n} = \vec{u} \times \vec{v}$  normal to both  $\vec{u}$  and  $\vec{v}$ .



PLANE THROUGH 3 POINTS  $P, Q, R$ : The vector  $(a, b, c) = \vec{n} = (Q - P) \times (R - P)$  is normal to the plane. Therefore, the equation is  $ax + by + cz = d$ . The constant is  $d = ax_0 + by_0 + cz_0$  because the point  $P = (x_0, y_0, z_0)$  is on the plane.

PLANE THROUGH POINT  $P$  AND LINE  $\vec{r}(t) = Q + t\vec{u}$ . The vector  $(a, b, c) = \vec{n} = \vec{u} \times (Q - P)$  is normal to the plane. Therefore the plane is given by  $ax + by + cz = d$ , where  $d = ax_0 + by_0 + cz_0$  and  $P = (x_0, y_0, z_0)$ .

LINE ORTHOGONAL TO PLANE  $ax + by + cz = d$  THROUGH POINT  $P$ . The vector  $\vec{n} = (a, b, c)$  is normal to the plane. The line is  $\vec{r}(t) = P + \vec{n}t$ .

ANGLE BETWEEN PLANES. The angle between the two planes  $a_1x + b_1y + c_1z = d_1$  and  $a_2x + b_2y + c_2z = d_2$  is  $\arccos(\vec{n}_1 \cdot \vec{n}_2 / (|\vec{n}_1| |\vec{n}_2|))$ , where  $\vec{n}_i = (a_i, b_i, c_i)$ . Alternatively, it is  $\arcsin(|\vec{n}_1 \times \vec{n}_2| / (|\vec{n}_1| |\vec{n}_2|))$ .

INTERSECTION BETWEEN TWO PLANES. Find the line which is the intersection of two non-parallel planes  $a_1x + b_1y + c_1z = d_1$  and  $a_2x + b_2y + c_2z = d_2$ . Find first a point  $P$  which is in the intersection. Then  $\vec{r}(t) = P + t(\vec{n}_1 \times \vec{n}_2)$  is the line, we were looking for.

### LINES IN THE PLANE.

LINE  $P = Q + t\vec{v}$ . Eliminating  $t$  gives a single equation. For example,  $(x, y) = (1, 2) + t(3, 4)$  is equivalent to  $x = 1 + 3t, y = 2 + 4t$  and  $4x - 3y = -2$ . The general equation for a line in the plane is  $ax + by = d$ .