

CROSS PRODUCT II

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BINET CAUCHY IDENTITY. Here is an identity

$$(axb) \cdot (cxd) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c).$$

You can either challenge yourself to verify it by hand or try to verify it with Mathematica

$a = \{x, y, z\}$; $b = \{u, v, w\}$; $c = \{o, p, q\}$; $d = \{r, s, t\}$;

Cross $[a, b] \cdot$ Cross $[c, d] = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$;

Simplify [%]

Here are some more historical additions to the notion of the cross product.

CROSS PRODUCT IN 2D. In two dimensions, there is a cross product too, but it is a scalar

$$\langle a, b \rangle \cdot \langle c, d \rangle = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

2D VECTORS AND COMPLEX NUMBERS: There is a relation when multiplying a complex number with the complex conjugate of another complex number. The crossed product in 2D is related with the complex multiplication.

$$z \cdot w = (a + ib) \cdot (u + iv) = au - bv - i(av + bu) = z \times w - i(z \cdot w)$$

3D VECTORS AND QUATERNIONS. It is historically interesting that it took for mathematicians a very long time to develop the cross product in three dimensions. It seems that it came from a different direction. The multiplication of quaternions with zero real parts

$$z \cdot w = (ia + jb + kc) \cdot (iu + jv + kw) = z \times w - z \cdot w$$

involves both the dot and cross product of $A = (a, b, c)$ and $U = (u, v, w)$. We can use this relation to define the product of two quaternions.

It might have been the path in which **William Hamilton** defined first a cross product. He did so by inventing something far more complicated, the Quaternion algebra.



THE CROSS PRODUCT IN 4 DIMENSIONS.

The cross product of two vectors in R^4 is a vector in R^6 . Writing the coordinates explicitly down depends on the chosen basis: one possibility is

$$(a, b, c, d) \times (t, u, v, w) = (au - bt, av - ct, aw - dt, bw - du, cu - bv, cw - dv)$$

so that the first three coordinates are $a(u, v, w) - t(b, c, d)$ and the last three coordinates are $(b, c, d) \times (u, v, w)$.

The cross product of two vectors in n dimensions is a vector in dimension $\binom{n}{2}$. This is rather a topic for multi-linear algebra and when dealing with multivariable calculus in higher dimensions.

LINES

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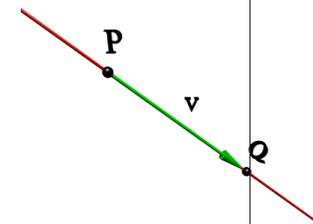
LINES. A point P and a vector \vec{v} define a line L . It is the set of points

$$L = \{P + t\vec{v}, \text{ where } t \text{ is a real number}\}$$

The line contains the point P and points into the direction \vec{v} .

EXAMPLE. $L = \{(x, y, z) = (1, 1, 2) + t(2, 4, 6)\}$.

This description is called the **parametric equation** for the line.



EQUATIONS OF LINE. We can write $(x, y, z) = (1, 1, 2) + t(2, 4, 6)$ so that $x = 1 + 2t, y = 1 + 4t, z = 2 + 6t$. If we solve the first equation for t and plug it into the other equations, we get $y = 1 + (2x - 1), z = 2 + 3(2x - 1)$. We can therefore describe the line also as

$$L = \{(x, y, z) \mid y = 2x - 1, z = 6x - 4\}$$

SYMMETRIC EQUATION. The line $\vec{r} = P + t\vec{v}$ with $P = (x_0, y_0, z_0)$ and $\vec{v} = (a, b, c)$ satisfies the **symmetric equations**

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

Proof. Every of the three expression is equal to t .

PROBLEM. Find the equations for the line through the points $P = (0, 1, 1)$ and $Q = (2, 3, 4)$.

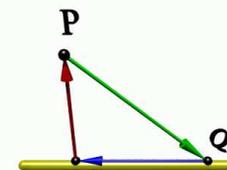
SOLUTION. The parametric equations are $(x, y, z) = (0, 1, 1) + t(2, 2, 3)$ or $x = 2t, y = 1 + 2t, z = 1 + 3t$. Solving each equation for t gives the symmetric equations $x/2 = (y - 1)/2 = (z - 1)/3$.

DISTANCE POINT-LINE (3D). If P is a point in space and L is the line $\vec{r}(t) = Q + t\vec{u}$, then

$$d(P, L) = |(P - Q) \times \vec{u}| / |\vec{u}|$$

is the distance between P and the line L .

This formula is verified by writing $(P - Q) \times \vec{u} = |P - Q| |\vec{u}| \sin(\theta)$.



EXAMPLE. $P = (1, 3, 1)$ is a point in space and L is the line $\vec{r}(t) = (1, 1, 1) + t(1, 0, 1)$. Then

$$d(P, L) = |(0, 2, 0) \times (1, 0, 1)| / \sqrt{2} = |(2, 0, -2)| / \sqrt{2} = \sqrt{8} / \sqrt{2} = 2$$

is the distance between P and L .