LINE INTEGRALS **Maths21a,O. Knill**

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2D: If $F(x,y)$ is a vector field in the plane and $\gamma : t \mapsto \vec{r}(t)$ is a curve, then $\boxed{\int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t)\, dt}$ is called the **line integral** of F along the curve γ.

3D: If $F(x,y,z)$ is a vector field in space and $\gamma : t \mapsto \vec{r}(t)$ is a curve, then $\boxed{\int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t)\, dt}$ is called the **line integral** of F along the curve γ.

NOTATION. The short-hand notation $\int_\gamma F \ \cdot ds$ is also used. In the literature, where curves are sometimes written as $r(t) = (x(t), y(t), z(t))$ or $r(t)$, the notation $\int_\gamma F \ \cdot dr$ or $\int_\gamma F \ \cdot dr$ appears. For simplicity, we leave out below the arrows above the $r(t)$ and $F(r(t))$ even so they are vectors.

WRITTEN OUT. If $F = (P,Q)$ and $\vec{r} = (x(t), y(t))$, we can write $\int_a^b P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)\, dt$.

MORE NOTATION. One also can write $\int_a^b P(x,y)\, dx + Q(x,y)\, dy$. Warning: this later notation is only possible for certain type of curves and rather old fashioned. Avoid it.

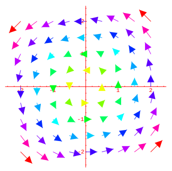
EXAMPLE: **Work**. If $F(x,y,z)$ is a force field, then the line integral $\int_a^b F(r(t)) \cdot r'(t)\, dt$ is called **work**.

EXAMPLE: **Electric potential**. If $E(x,y,z)$ is an electric field, then the line integral $\int_a^b E(r(t)) \cdot r'(t)\, dt$ is called **electric potential**.

EXAMPLE: **Gradient field**. If $F(x,y,z) = \nabla U(x,y,z)$ is a gradient field, then as we will see next hour $\int_a^b F(r(t)) \cdot r'(t)\, dt = U(r(b)) - U(r(a))$. The gradient field has physical relevance. For example, if $U(x,y,z)$ is the pressure distribution in the atmosphere, then $\nabla U(x,y,z)$ is the pressure gradient roughly the wind velocity field.

EXAMPLE 1. Let $\gamma : t \mapsto r(t) = (\cos(t), \sin(t))$ be a circle parametrized by $t \in [0, 2\pi]$ and let $F(x,y) = (-y, x)$. Calculate the line integral $I = \int_\gamma F(r) \cdot dr$.

ANSWER: We have $I = \int_0^{2\pi} F(r(t)) \cdot r'(t)\ dt = \int_0^{2\pi} (-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t))\, dt = \int_0^{2\pi} \sin^2(t) + \cos^2(t)\, dt = 2\pi$



EXAMPLE 2. Let $r(t)$ be a curve given in polar coordinates as $r(t) = \cos(t), \phi(t) = t$ defined on $[0, \pi]$. Let F be the vector field $F(x,y) = (-xy, 0)$. Calculate the line integral $\int_\gamma F \cdot dr$.

SOLUTION. In Cartesian coordinates, the curve is $r(t) = (\cos^2(t), \cos(t)\sin(t))$. The velocity vector is then $r'(t) = (-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t)) = (x(t), y(t))$. The line integral is

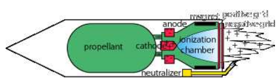
$$\int_0^\pi F(r(t)) \cdot r'(t)\, dt = \int_0^\pi (\cos^3(t)\sin(t), 0) \cdot (-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t))\, dt$$

$$= -2\int_0^\pi \sin^2(t)\cos^4(t)\, dt = -2(t/16 + \sin(2t)/64 - \sin(4t)/64 - \sin(6t)/192)|_0^\pi = -\pi/8$$

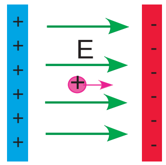
WORK. If F is a **force field** and $\vec{r}(t)$ a path of a body, then $F(r(t))$ is the force acting on the body. The component of that force in the velocity direction is $G(t) = F(r(t)) \cdot r'(t)/|r'(t)|$. For some small time dt, the body will move a distance $|r'(t)|dt$. In physics, $G(t)ds$ is the amount of work done when traveling this distance. Integrating up gives the total work or energy $W = \int_a^b G(t)|r'(t)|\, dt = \int_a^b = F(r(t)) \cdot r'(t)\, dt$.

$W = \int_\gamma F\, ds$ is the **energy** gained by a body traveling along the path γ in a force field F.

EXAMPLE. In **ion rockets** (used for example in "deep space" space craft), ionized xenon gas is passed by an electrically charged plate and accelerated to high velocities 30km/s. This (kinetic) energy is built up as work in a force field F which is parallel to an electric field E. Let $\gamma : r(t) = (t, 0, 0), 0 \leq t \leq L$ be the path a particle travels between the positively charged plate and the negatively charged plate.



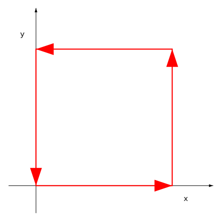
$\int_\gamma E\ dr$ is the voltage difference between the two plates and $\int_\gamma F\ dr$ is the energy difference of the particle. Because $F = cE$, where e is the charge of the ion and the velocity r' of the ions is parallel to the field, we know that ELc is the voltage difference and FL is the energy difference which is $mv^2/2$, where m is the mass and v the velocity of the ion, we could get the electric field strength $E = mv^2/(2Lc)$.



ADDING AND SUBTRACTING CURVES.

If γ_1, γ_2 are curves, then $\gamma_1 + \gamma_2$ denotes the curve obtained by traveling first along γ_1, then along γ_2. One writes $-\gamma$ for the curve γ traveled backwards and $\gamma_1 - \gamma_2 = \gamma_1 + (-\gamma_2)$.

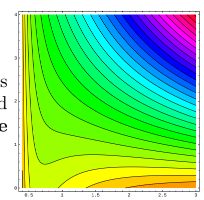
EXAMPLES. If $\gamma_1(t) = (t, 0)$ for $t \in [0,1]$, $\gamma_2(t) = (1, (t-1))$ for $t \in [1,2]$, $\gamma_3(t) = (1-(t-2), 1)$ for $t \in [2,3]$, $\gamma_4(t) = (0, 1-(t-3))$ for $t \in [3,4]$, then $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ for $t \in [0,4]$ is the path which goes around a the unit square. The path $-\gamma$ travels around in the clockwise direction.



CALCULATING WITH LINE-INTEGRALS.

- $\int_\gamma F \cdot dr + \int_\gamma G \cdot dr = \int_\gamma (F+G) \cdot dr$.
- $\int_{\gamma_1+\gamma_2} F \cdot dr = \int_{\gamma_1} F \cdot dr + \int_{\gamma_2} F \cdot dr$
- $\int_\gamma cF \cdot dr = c\int_\gamma F \cdot dr$.
- $\int_{-\gamma} F \cdot dr = -\int_\gamma F \cdot dr$.

VOLUME-PRESSURE. Processes involving gases or liquids can be described in a **Volume-pressure diagram**.



Left: V-P diagram
Right: Sadi Carnot

A periodic processes like a refrigerator defines a closed cycle $\gamma : t \mapsto r(t) = (V(t), p(t))$ in the $V-p$ plane. The curve is parameterized by the time t. At a given time the gas has a specific volume $V(t)$ and a specific pressure $p(t)$. Consider the vector field $F(V,p) = (p, 0)$ and a closed curve γ and the line integral $\int_\gamma F\ ds$. Writing it out, we get $\int_0^{2\pi} (p(t), 0) \cdot (V'(t), p'(t))\, dt = \int_0^{2\pi} p(t)V'(t)\, dt = \int_0^{2\pi} p\ dV$.

If the volume of the gas changes under pressure p, then the work on the system is pdV. On the other hand, if the volume is kept constant, then for a gas, one does not do work on the system, when changing the pressure. Processes described by this approximation are called **adiabatic**.

For example, if the volume is decreased under high pressure and increased under low pressure then we do the work $\int_0^{2\pi} p\ dV$. Lets compute that if $r(t) = (2+\cos(t), 2+\sin(t))$ for $t \in [0, 2\pi]$ and $F(V,p) = (p, 0)$. $r'(t) = (-\sin(t), \cos(t))$, $F(r(t)) = (2+\sin(t), 0)$. so that $F(r(t)) \cdot r'(t) = -\sin(t)(2+\sin(t))$ and $\int_0^{2\pi} F(r(t)) \cdot r'(t)\, dt = -\int_0^{2\pi} \sin^2(t)\, dt = \pi$.