This is part 3 (of 3) of the weekly homework. It is due August 5'th in class.

SUMMARY.

- $\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)dt$ **line integral** of F along curve $C : t \mapsto r(t)$.
- Example: $C : r(t) = (\cos(t), \sin(t)), t \in [0, 2\pi]$ (circle), $F(x, y) = (-y, x)$. $\int_C F \cdot dr = \int F(r(t)) \cdot r'(t)\, dt = \int_0^{2\pi}(-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t))\, dt = \int_0^{2\pi} 1\, dt = 2\pi$.
- We say F **is conservative** or F **is a gradient field** if $F(x, y, z) = \nabla f(x, y, z)$.
- $\int_C \nabla f\ dr = f(r(b)) - f(r(a))$ **fundamental theorem of line integrals**: by chain rule: $\int_a^b \nabla f(r(t)) \cdot r'(t)\, dt = \int_a^b \frac{d}{dt} f(r(t))\, dt$. Apply the fundamental theorem of calculus.
- If F is conservative in the plane then the line integrals do not depend on the path.
- If F is conservative in the plane and C is a closed curve, then $\int_C F\ dr = 0$.

Homework Problems

1) (4 points) Find a closed curve $C : \vec{r}(t)$ for which the vector field $F(x, y) = (P(x, y), Q(x, y)) = (xy, x^2)$ satisfies $\int_C F(r(t)) \cdot r'(t)\, dt \neq 0$.

Solution:
We knew already that, because $Q_x = 2x$ and $P_y = x$, the field can not be a gradient field $(P, Q) = (f_x, f_y)$. Any curve not symmetric with respect to the y axes should work, for example a circle centered at $(1, 1)$.

2) (4 points) Let C be the circle $x^2 + y^2 = 16$ and $F(x, y) = (x, y^4)$. Calculate the line integral $\int_C F \cdot dr$.

Solution:
Actually, this is a conservative field because $F(x, y) = \nabla f(x, y)$ with $f(x, y) = x^2/2 + y^5/5$ so that the line integral is zero.
However, we can also compute the line integral: $\vec{r}(t) = (4\cos(t), 4\sin(t))$ parametrizes the circle. $\vec{r}'(t) = (-4\sin(t), 4\cos(t))$ and $F(r(t)) = (4\cos(t), 256\sin^4(t))$ so that $\int_0^{2\pi} F(r(t)) \cdot r'(t)\, dt = \int_0^{2\pi}(-16\sin(t)\cos(t) + 1024\sin^4(t)\cos(t))\, dt = (8\sin^2(t) + 1024\sin^5(t)/5t)|_0^{2\pi} = 0$.

3) (4 points) Let C be the space curve $\vec{r}(t) = (\cos(t), \sin(t), t)$ for $t \in [0, 1]$ and let $F(x, y, z) = (y, x, 5)$. Calculate the line integral $\int_C F \cdot dr$.

Solution:
$\int_C f \cdot dr = \int_0^1(\sin(t), \cos(t), 5) \cdot (-\sin(t), \cos(t), 1)\, dt = \int_0^1(\cos(2t) + 5)\, dt = \sin(2)/2 + 5$.

4) (4 points) Find the work done by the force field $F(x, y) = (x\sin(y), y)$ on a particle that moves along the parabola $y = x^2$ from $(-1, 1)$ to $(2, 4)$.

Solution:
We parametrize the curve C by $r(t) = (t, t^2)$ so that $r'(t) = (1, 2t)$.
$W = \int_C F \cdot dr = \int_{-1}^2 (t\sin(t^2), t^2) \cdot (1, 2t)\, dt = \int_{-1}^2 (t\sin(t^2) + 2t^3)\, dt = \cos(1)/2 - \cos(4)/2 + 15/2$.

5) (4 points) Let F be the vector field $F(x, y) = (-y, x)/2$. Compute the line integral of F along an ellipse $\vec{r}(t) = (a\cos(t), b\sin(t))$ with width $2a$ and height $2b$. The result should depend on a and b.

Solution:
The velocity is $\vec{r}'(t) = (-a\sin(t), b\cos(t))$ and $F(\vec{r}(t)) = (-b\sin(t), a\cos(t))/2$ so that $F(\vec{r}(t)) \cdot \vec{r}'(t) = ab/2$. If we integrate this from 0 to 2π we get the result πab.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) Consider a O shaped pipe which is filled only on the right side with water. A wooden ball falls on the right hand side in the air and moves up in the water. Why does this "perpetual motion machine" not work?

2) What is wrong with the Escher pictures with the stair in which people always walk down or with the waterfall. The figures suggests the existence of a force field which is not conservative. How do the Escher pictures "work"?