This is part 1 (of 3) of the weekly homework. It is due July 22 at the beginning of class.

SUMMARY.

- $\nabla f(x,y,z) = (f_x, f_y, f_z)$ **gradient**.
- $D_v f = \nabla f \cdot v$ **directional derivative**.
- $f(x,y,z)$ function of three variables, $r(t)$ **curve**, $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ **chain rule**.
- $n \cdot (x,y,z) = d = n \cdot (x_0,y_0,z_0)$ **tangent plane** to $f(x,y,z) = c$ at point (x_0,y_0,z_0).
- $\nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = 0$ **tangent plane** to $f(\vec{x}) = c = f(\vec{x}_0)$ (vector notation).
- "Gradients are orthogonal to level curves rsp. level surfaces."
- $L(x,y) = f(x_0,y_0) + a(x-x_0) + b(y-y_0)$ **linear approximation** of $f(x,y)$ at (x_0,y_0).
- **Tangent line** $ax + by = d$ with $a = f_x(x_0,y_0), b = f_y(x_0,y_0)$, $d = ax_0 + by_0$.
- **Estimate** $f(x,y)$ by $L(x,y)$ near $f(x_0,y_0)$.
- Vector notation: $L(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0)$
- Punch line: $L(\vec{x})$ is close to $f(\vec{x})$ near $\vec{x}_0$ but a simple linear function. Many physical laws are actually linear approximations to more complicated laws.

Homework Problems

1) (4 points)
a) Sketch a contour map of the function $f(x,y) = x^2 + 9y^2$.
b) Find the **gradient vector** $\nabla f = (f_x, f_y)$ of f at the point $(1,1)$ and draw it.
c) Find the tangent line $ax + by = d$ to the curve at $(1,1)$ and draw it.
d) Estimate $f(1.001, 0.999)$ using linear approximations.

2) (4 points) The pressure in the space at the position (x,y,z) is $p(x,y,z) = x^2 + y^2 - z^3$ and the trajectory of an observer is the curve $\vec{r}(t) = \langle t, t, 1/t \rangle$.
a) (2 points) State the chain rule which applies in this situation.
b) (2 points) Using the chain rule in a) compute the rate of change of the pressure the observer measures at time $t = 2$.

3) (4 points)
Suppose $2x + 3y + 2z = 9$ is the tangent plane to the graph of $z = f(x,y)$ at the point $(1,1,2)$.

a) (2 points) What is the linear approximation of $f(1.01, 0.98)$? b) (1 point) What is the gradient ∇f at $(1,1)$?
c) (1 point) What is the equation $ax + by = d$ of the tangent line at $(1,1)$?

4) (4 points)
a) (2) Find the linear approximation $L(x,y)$ of the function $f(x,y) = \sqrt{10 - x^2 - 5y^2}$ at $(2,1)$ and use it to estimate $f(1.95, 1.04)$.
b) (2) Find the directional derivative $D_v f(2,1) = \nabla f(2,1) \cdot \vec{v}$ into the direction $\vec{v} = (-3,4)/5$.

5) (4 points)
Find $f(0.01, 0.999)$ for $f(x,y) = \cos(\pi x y) y + \sin(x + \pi y)$

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) When is the estimate for $\sqrt{n}$ using linear approximations the worst? What is the maximal error when computing square roots using linearlization? For example, to compute $\sqrt{102}$, we estimate this with linear approximation as $10 + 2/20 = 10.1$, while the true result is 10.09950493836207795....

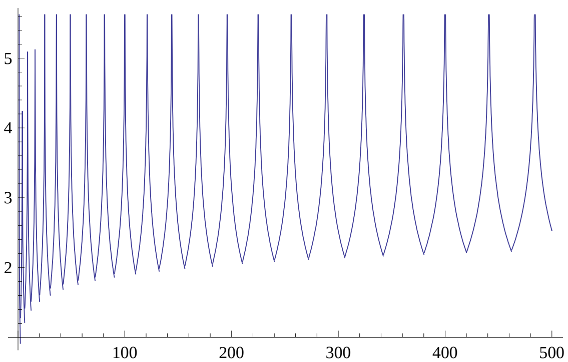

Figure: the number of decimal places which are correct with the linear estimate of the square root.

2) Marsden and Tromba pose in their textbook the following riddle:
Suppose $w = f(x,y)$ and $y = x^2$. By the chain rule

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + 2x\frac{\partial w}{\partial y}$$

so that $0 = 2x\frac{\partial w}{\partial y}$ and so $\frac{\partial w}{\partial y} = 0$. Find an explicit example of a function $f(x,y)$, where you see the argument is false.

3) The partial derivatives of the function $f(x,y) = (xy)^{1/3}$ exists at every point but the directional derivatives in all other directions don't exist at the point $(0,0)$. What is going on?

4) Extend the notion of "tangent plane" to 3-dimensional hyper-surfaces $f(x,y,z,w) = c$ in 4-dimensional space. For example, what is the tangent plane to the three-dimensional sphere $x^2 + y^2 + z^2 + w^2 = 1$ at the point $(x,y,z,w) = (1/2, 1/2, 1/2, 1/2)$?