6/26/2008 CROSS PRODUCT/LINES **O.Knill,Maths21a**

This is part 3 (of 3) of the homework which is due Tuesday July 1 at the beginning of class.

SUMMARY.

- $\vec{v} \times \vec{w} = \langle v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1 \rangle = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$. **cross product**.
- $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin(\alpha)$, where α is the **angle** between vectors. **Area** of parallelogram spanned by $\vec{v}$ and $\vec{w}$.
- $\vec{v} \times \vec{w}$ is **orthogonal** to $\vec{v}$ and to $\vec{w}$ with length $|\vec{v}||\vec{w}|\sin(\phi)$
- $\vec{u} \cdot (\vec{v} \times \vec{w})$ **triple scalar product**, signed volume of parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$.
- $\vec{r}(t) = P + t\vec{v}$ **parametric equation** for a line, P a point, $\vec{v}$ is a vector.
- $\frac{(x-x_0)}{a} = \frac{(y-y_0)}{b} = \frac{(z-z_0)}{c}$ **symmetric equation** for a line.
- Distance Point-Line $d(P, L) = |(\vec{PQ}) \times \vec{u}|/|\vec{u}|$.

Homework Problems

1) (4 points)
a) (2) Find a the cross product $\vec{w}$ of $\vec{u} = (-3, -1, 2)$ and $\vec{v} = (-2, -2, 5)$.
b) (1) Find a unit vector $\vec{n}$ orthogonal to $\vec{u}$ and $\vec{v}$.
c) (1) Find the volume of the parallelepiped spanned by $\vec{u}, \vec{v}$ and $\vec{w}$.

Solution:
a) $\vec{u} \times \vec{v} = \vec{w} = (-1, 11, 4)$
b) $|\vec{w}| = \sqrt{138}$, $\vec{n} = \vec{w}/|\vec{w}| = (-1/\sqrt{138}, 11/\sqrt{138}, 4/\sqrt{138})$.
c) $\vec{w} \cdot (\vec{u} \times \vec{v}) = \vec{w} \cdot \vec{w} = |\vec{w}|^2 = 138$.

2) (4 points)
a) (2) Find the distance between the point $P = (2, -1, 2)$ and the line $x = y = z$.
b) (2) Find a parametrization $r(t)$ of the line given in $a)$ and find the minimum of the function $f(t) = d(P, r(t))$. Check whether the minimal value is the distance you got in a).

3) (4 points)
a) Assume $\vec{u} + \vec{v} + \vec{w} = \vec{0}$. Verify that $\vec{u} \times \vec{v} = \vec{v} \times \vec{w} = \vec{w} \times \vec{u}$.
b) Find $(\vec{u} + \vec{v}) \cdot (\vec{v} \times \vec{w})$ if $\vec{u}, \vec{v}, \vec{w}$ are unit vectors which are orthogonal to each other and $\vec{u} \times \vec{v} = \vec{w}$.
c) Assume you have a triangle in the plane which has edge points with integer coordinates. Show that the area of the triangle is an integer, or half of an integer.

Solution:
a) Build a triangle with the three vectors u, v, w. Each of the terms in the identity is twice the area of the triangle.
b) We have $(\vec{u} + \vec{v}) \cdot (\vec{v} \times \vec{w}) = (\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{w} \cdot (\vec{u} \times (\vec{v}) = \vec{w} \cdot \vec{w} = 1$.
c) The formula for the area is $|(A - C) \times (B - C)|/2$.

4) (4 points) a) Verify the identity $\vec{b} \times (\vec{u} \times \vec{v}) = (\vec{b} \cdot \vec{v})\vec{u} - (\vec{b} \cdot \vec{u})\vec{v}$.
b) (2 points) Verify that in general, if $\vec{u} \cdot (\vec{v} \times \vec{w}) = 1$ and

$$\begin{aligned} \vec{a} &= \vec{v} \times \vec{w} \\ \vec{b} &= \vec{w} \times \vec{u} \\ \vec{c} &= \vec{u} \times \vec{v} \, , \end{aligned}$$

then $\vec{a} \cdot (\vec{b} \times \vec{c}) = 1$.

Solution:
a) This is a direct computation.
b) Focus first on $(\vec{b} \times \vec{c}) = \vec{b} \times (\vec{u} \times \vec{v})$ and use a) to get $(\vec{b} \cdot \vec{v})\vec{u} - (\vec{b} \cdot \vec{u})\vec{v}$. The problem asks for the dot product of this with $\vec{a}$. Now, since $\vec{a}$ is orthogonal to the second term, we obtain $\vec{a}(\vec{b} \cdot \vec{v}) \cdot \vec{u}$. When plugging in the definitions of $\vec{a}$ and $\vec{b}$, we are left with 1.

5) (4 points)
a) (2) Find the parametric and symmetric equation for the line which passes through the points $P = (1, 2, 3)$ and $Q = (3, 4, 5)$.
b) (2) Find the equation for the plane which contains the three points $P = (1, 2, 3), Q = (3, 4, 4)$ and $R = (1, 1, 2)$. This problem is a preparation for next week. You can use without proof that a plane has the form $ax + by + cz = d$ where $\vec{n} = \langle a, b, c \rangle$ is thew normal vector to the plane.

Solution:
a) The vector $\vec{v} = (2, 2, 2)$ connects the two points. The parametric equation is $P + t\vec{v} = (1, 2, 3) + t(2, 2, 2) = (1 + 2t, 2 + 2t, 3 + 2t)$. The symmetric equation is $(x - 1)/2 = (y - 2)/2 = (z - 3)/2$.
b) A normal vector $\vec{n} = (1, -2, 2) = (a, b, c)$ of the plane $ax + by + cz = d$ is obtained as the cross product of $P - Q$ and $R - Q$ With $d = \vec{n} \cdot P = 3$, we have the equation $x - 2y + 2z = 3$.

Remarks

(You don't need to read these remarks to do the problems.)

To 4): three vectors with nonzero triple scalar product are called **non-coplanar** . Adding integer multiples of such vectors form a **lattice** $n\vec{u} + m\vec{v} + k\vec{w}$, where n, m, k are integers. The vectors $\vec{a}, \vec{b}$ and $\vec{c}$ in the problem define a new lattice which is called the **reciprocal lattice**. Crystallographers also denote them by $\vec{u}^*, \vec{v}^*$ and $\vec{w}^*$. The reciprocal lattice is essential for the study of crystal lattices and their diffraction properties obtained by shooting X-rays onto them.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) Prove the following identity for vectors $\vec{a}, \vec{b}, \vec{c}$ in space:

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \, .$$

Or the Binet-Cauchy identity

$$(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c) \, .$$

2) Find a general formula for the volume of a tetrahedron with corners P, Q, R, S.
Hint. Find first a formula for the area of one of its triangular faces, and then a formula for the distance from the fourth point to that face.

3) The change of the angular momentum satisfes the formula

$$\frac{d}{dt}\vec{L} = \vec{L} \times \vec{\Omega} ,$$

where $\vec{\Omega}$ is the angular velocity vector. Verify that the length of $\vec{L}$ does not change in time. To do so, compute the time derivative $\vec{L} \cdot \vec{L}$ and show that it is 0. You can use

$$\frac{d}{dt}[\vec{v}(t) \times \vec{w}(t)] = \vec{v}\,'(t) \times \vec{w}(t) + \vec{v}(t) \times \vec{w}\,'(t) .$$