Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Provide details to all computations except for problems 1-3.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

| | | |
|---|---|---|
| 1 | | 20 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| Total: | | 100 |

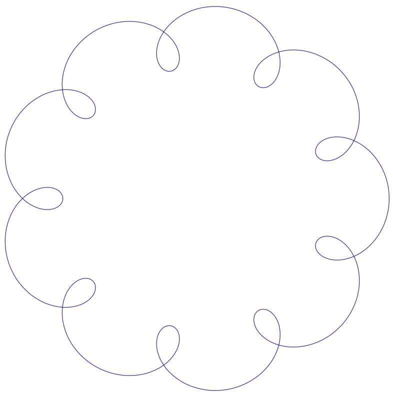
| Problem 1) True/False questions (20 points) |
|---|

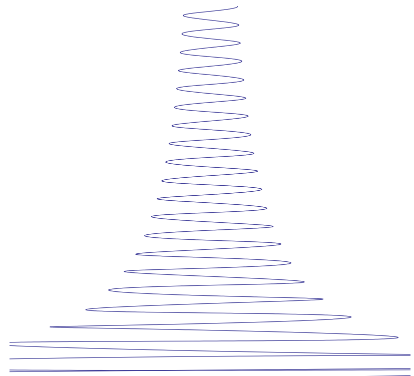
Mark for each of the 20 questions the correct letter. No justifications are needed.

| | | | |
|---|---|---|---|
| 1) | T | F | The cosine of the angle between the velocity vector and the acceleration vector is always positive. |
| 2) | T | F | The function $f(t,x) = x^3 - 3xy^2$ satisfies the Laplace partial differential equation $f_{xx} + f_{yy} = 0$. |
| 3) | T | F | The velocity vector of the curve $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ at time $t = 0$ is perpendicular to the curve at $\vec{r}(0)$. |
| 4) | T | F | If every point of the plane is a critical point for a function $f$ then $f$ is a constant function. |
| 5) | T | F | It is possible that $(1,1)$ is a local maximum for the function $f$ and $1 = f_{xx} = -f_{yy}$. |
| 6) | T | F | $(0,0)$ is a local maximum of the function $f(x,y) = 5 - x^8 - y^8$. |
| 7) | T | F | If $f(x,y)$ has a local max at the point $(0,0)$ with discriminant $D > 0$, then $g(x,y) = f(x,y) - x^9 + y^9$ has a local max at $(0,0)$ too. |
| 8) | T | F | The value of the function $f(x,y) = \log(e + 3x + 5y)$ at $(-0.002, 0.01)$ can by linear approximation be estimated as $1 - 0.006 + 0.005$. |
| 9) | T | F | The curvature of the curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle = (5, 3\sin(t), 3\cos(t))$ is $1/3$. |
| 10) | T | F | The chain rule tells that $\frac{d}{dt} f(\vec{r}(t)) = f(\vec{r}(t)) r'(t)$ |
| 11) | T | F | If the curvature of a curve is zero everywhere, then it is a line. |
| 12) | T | F | The gradient of $f$ at a point $(x_0, y_0, z_0)$ is orthogonal to the level surface of $f$ which contains $(x_0, y_0, z_0)$. |
| 13) | T | F | If the Lagrange multiplier $\lambda$ is negative then the critical point under constraint is a saddle point. |
| 14) | T | F | An aroplane has a velocity vector which happens to coincide with the gradient vector of the pressure at that point. The rate of change of the pressure is positive. |
| 15) | T | F | The arc length of a curve on $[0,1]$ can be obtained by integrating up the curvature of the curve along the interval $[0,1]$. |
| 16) | T | F | If $D$ is the discriminant at a critical point and $D f_{xx} > 0$ then we either have a saddle point or a local maximum. |
| 17) | T | F | The function $f(x,y) = \sin(y) x^2 \sin(y^2)$ satisfies the partial differential equation $f_{xyyxyxy} = 0$. |
| 18) | T | F | If $(0,0)$ is a critical point of $f(x,y)$ and the discriminant $D$ is zero but $f_{xx}(0,0) > 0$ then $(0,0)$ can not be a local maximum. |
| 19) | T | F | In the second derivative test, one can replace the condition $D > 0, f_{xx} > 0$ with $D > 0, f_{xy} > 0$ to check whether a point is a local minimum. |
| 20) | T | F | If the curvature of a curve $\vec{r}(t)$ is equal to 1 everywhere and the curve connects the point $(0,0,0)$ with the point $(1,0,0)$, then the arc length is larger than 1. |

Problem 2) (10 points)

Match the parameterizations with the curves. No justifications are needed.


I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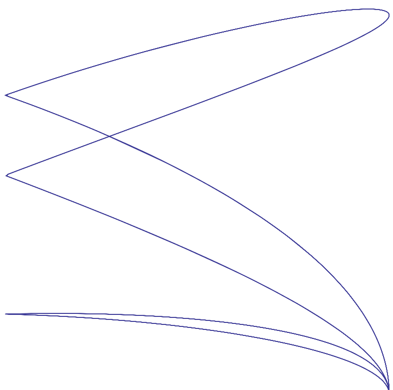
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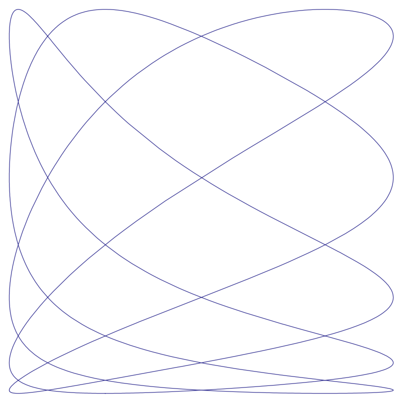
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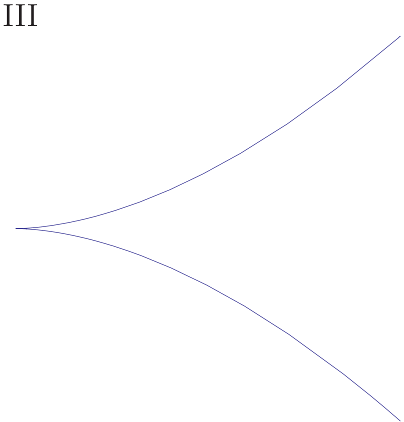
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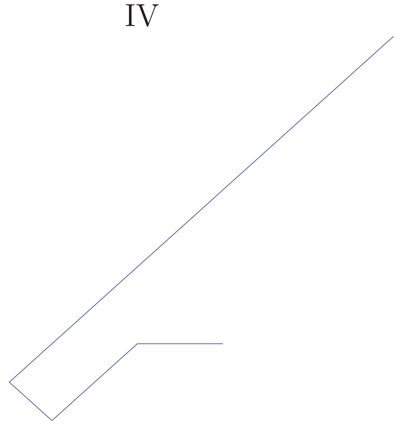

VI

| Enter I,II,III,IV here | Parameterization |
|---|---|
| | $\vec{r}(t) = \langle \lvert 1 - t + -3t \rvert, \lvert t - \lvert 1-t \rvert \rvert \rangle$ |
| | $\vec{r}(t) = \langle 1/(2+\sin(5t)), 1/(2+\cos(3t) \rangle$ |
| | $\vec{r}(t) = \langle \cos(t) + \cos(10t)/5, \sin(t) + \sin(10t)/5 \rangle$ |
| | $\vec{r}(t) = \langle \cos(7t)/t, \sin(9t)/t + 5t \rangle$ |
| | $\vec{r}(t) = \langle \lvert \cos(3t) \rvert, \lvert \sin(2t) + \sin(t) \rvert \rangle$ |
| | $\vec{r}(t) = \langle t^4, t^7 \rangle$ |

Problem 3) (10 points)

The following statements are not complete. Fill in from the pool of words below.

| statement | Fill in the letters | statement |
|---|---|---|
| The arc length does | | on the parametrization. |
| $\sqrt{48}$ can be estimated by | | at $x = 7$. The result is 7-1/14. |
| The velocity vector is | | to the curve. |
| The discriminant $D$ is | | if the point is a saddle point. |
| The unit normal vector is | | to the unit tangent vector. |
| For a Lagrange minimum, $\nabla g$ is | | to $\nabla f$. |
| The curvature does | | on the parametrization of the curve. |
| The trajectory of a ball is a | | if we are in free fall. |
| Arc length is approximated by a | | sum if the curve is smooth. |
| The gradient $\nabla f$ is | | to the surface $f = c$. |

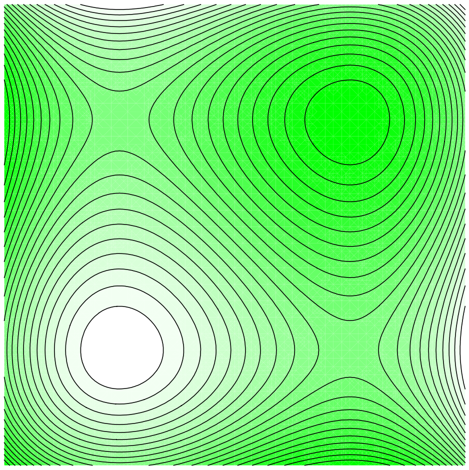
| | |
|---|---|
| B | negative |
| U | linear approximation |
| D | not depend |
| M | tangent |
| O | parabola |
| D | not depend |
| E | perpendicular |
| L | parallel |
| E | orthogonal |
| R | Rieman |

Problem 4) (10 points)

The green near one of the holes in the Cambridge Fresh pond golf course has the height

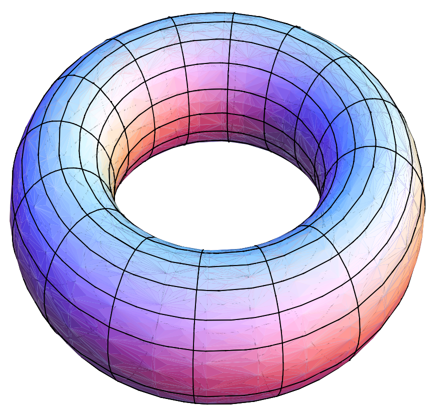
$$f(x, y) = x^3 + y^3 - 3x^2 - 3y^2$$

Find local maxima, local minima or saddle points of this function. Near which point will golf balls most likely end up, if balls like to roll to lower areas.




Problem 5) (10 points)

A torus can be obtained by rotating a circle of radius $b$ around a circle of radius $a$. The volume of such a torus is $2\pi^2 ab^2$ and the surface area is $4\pi^2 ab$. If we want to find the torus which has minimal surface area while the volume with fixed packing $2\pi^2 a(b^2+1)$ is fixed $2\pi^2$, we need to extremize the function $f(a,b) = 4\pi^2 ab$ under the constraint $a + ab^2 = 1$. Find the optimal $a, b$.



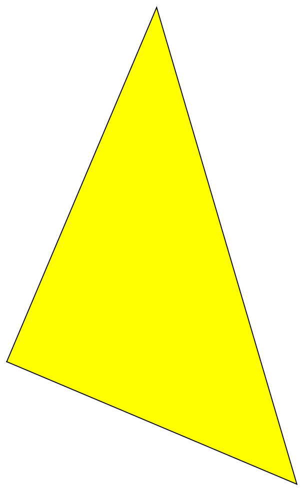
Problem 6) (10 points)

a) Find the arc length of the curve $\vec{r}(t) = \langle t^2, 2t^3/3, 1\rangle$ from $t = -1$ to $t = 1$.
b) What is the curvature of the curve at time $t = 1$? The formula for the curvature is

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

Problem 7) (10 points)

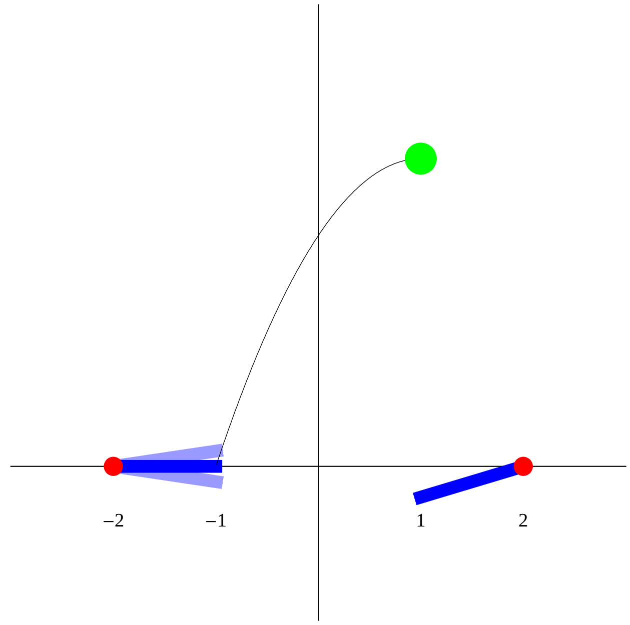
A right angle triangle has the side lengths $x = 0.999$ and $y = 1.00001$. Estimate the value of the hypotenuse $f(x, y) = \sqrt{x^2 + y^2}$ using linear approximation.



Problem 8) (10 points)

A pinball machine is tilted in such a way that a ball in the $xy$ plane experiences a constant force $\vec{F} = \langle 0, -2 \rangle$. A ball of mass 1 is hit the left flipper at the point $\vec{r}(0) = \langle -1, 0 \rangle$ with velocity $\vec{r}\,'(0) = \langle 1/2, 5 \rangle$.

a) Compute the trajectory $\vec{r}(t) = \langle x(t), y(t) \rangle$ of the ball.
b) The right flipper can reach the interval $[1, 2]$ on the $x$-axes. Can the player hit the ball when it comes back again to the base line $y = 0$?




| Problem 9) (10 points) |
|---|

Find the tangent plane to the surface

$$\sin(x+y) - \cos(z-x) + \sin(y) = -1$$

at the point $(0, \pi, 0)$.