Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work (the actual exam will have only 10 questions)

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
Total:		120

Problem 1) (20 points)

Circle for each of the 20 questions the correct letter. No justifications are needed.

1) T F The set of points in the plane which satisfy $x^2 - y^2 = -10$ is a curve called hyperbola.

2) T F The length of the sum of two vectors in space is always the sum of the length of the vectors.

3) T F For any three vectors $\vec{u}, \vec{v}, \vec{w}$, the identity $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{w} \times \vec{v}) \cdot \vec{u}$ holds.

4) T F The set of points which satisfy $x^2 + 2x + y^2 - z^2 = 0$ is a cone.

5) T F If P, Q, R are 3 different points in space that don't lie in a line, then $\vec{PQ} \times \vec{RQ}$ is a vector orthogonal to the plane containing P, Q, R.

6) T F The line $\vec{r}(t) = (1 + 2t, 1 + 3t, 1 + 4t)$ hits the plane $2x + 3y + 4z = 9$ at a right angle.

7) T F A surface which is given as $r = \sin(z)$ in cylindrical coordinates stays the same when we rotate it around the y axis.

8) T F For any two vectors, $\vec{v} \times \vec{w} = \vec{w} \times \vec{v}$.

9) T F If $|\vec{v} \times \vec{w}| = 0$ for all vectors $\vec{w}$, then $\vec{v} = \vec{0}$.

10) T F If $\vec{u}$ and $\vec{v}$ are orthogonal vectors, then $(\vec{u} \times \vec{v}) \times \vec{u}$ is parallel to $\vec{v}$.

11) T F Every vector contained in the line $\vec{r}(t) = (1 + 2t, 1 + 3t, 1 + 4t)$ is parallel to the vector $(1, 1, 1)$.

12) T F If in spherical coordinates a point is given by $(\rho, \theta, \phi) = (2, \pi/2, \pi/2)$, then its rectangular coordinates are $(x, y, z) = (0, 2, 0)$.

13) T F The set of points which satisfy $x^2 - 2y^2 - 3z^2 = 0$ form an ellipsoid.

14) T F If $\vec{v} \times \vec{w} = (0, 0, 0)$, then $\vec{v} = \vec{w}$.

15) T F The set of points in $\mathbf{R}^3$ which have distance 1 from a line form a cylinder.

16) T F If in rectangular coordinates, a point is given by $(1, 0, 1)$, then its spherical coordinates are $(\rho, \theta, \phi) = (\sqrt{2}, \pi/2, -\pi/2)$.

17) T F In spherical coordinates, the equation $\cos(\theta) = \sin(\theta)$ defines the plane $x - y = 0$.

18) T F For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, we always have $(\vec{a} \times \vec{b}) \cdot \vec{c} = -(\vec{a} \times \vec{c}) \cdot \vec{b}$.

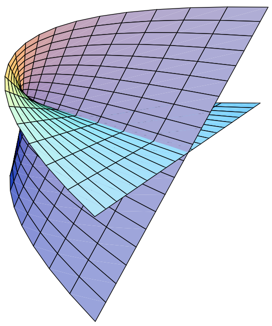
19) T F If $|\vec{v} \times \vec{w}| = 0$ then $\vec{v} = 0$ or $\vec{w} = 0$.

20) T F Two nonzero vectors are parallel if and only if their cross product is $\vec{0}$.

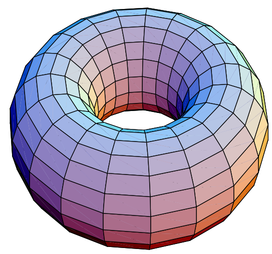
Problem 2a) (5 points)

Match the surfaces with their parameterization $\vec{r}(u, v)$ or the implicit description $g(x, y, z) = 0$. Note that one of the surfaces is not represented by a formula. No justifications are needed in this problem.

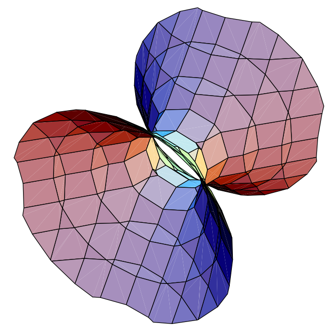
I



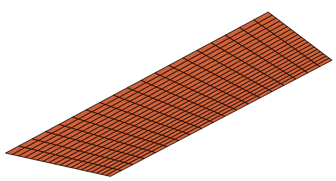
II



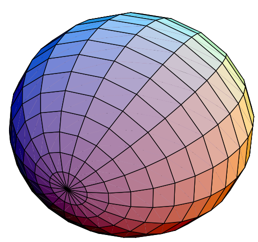
III



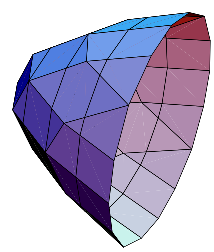
IV



V



VI



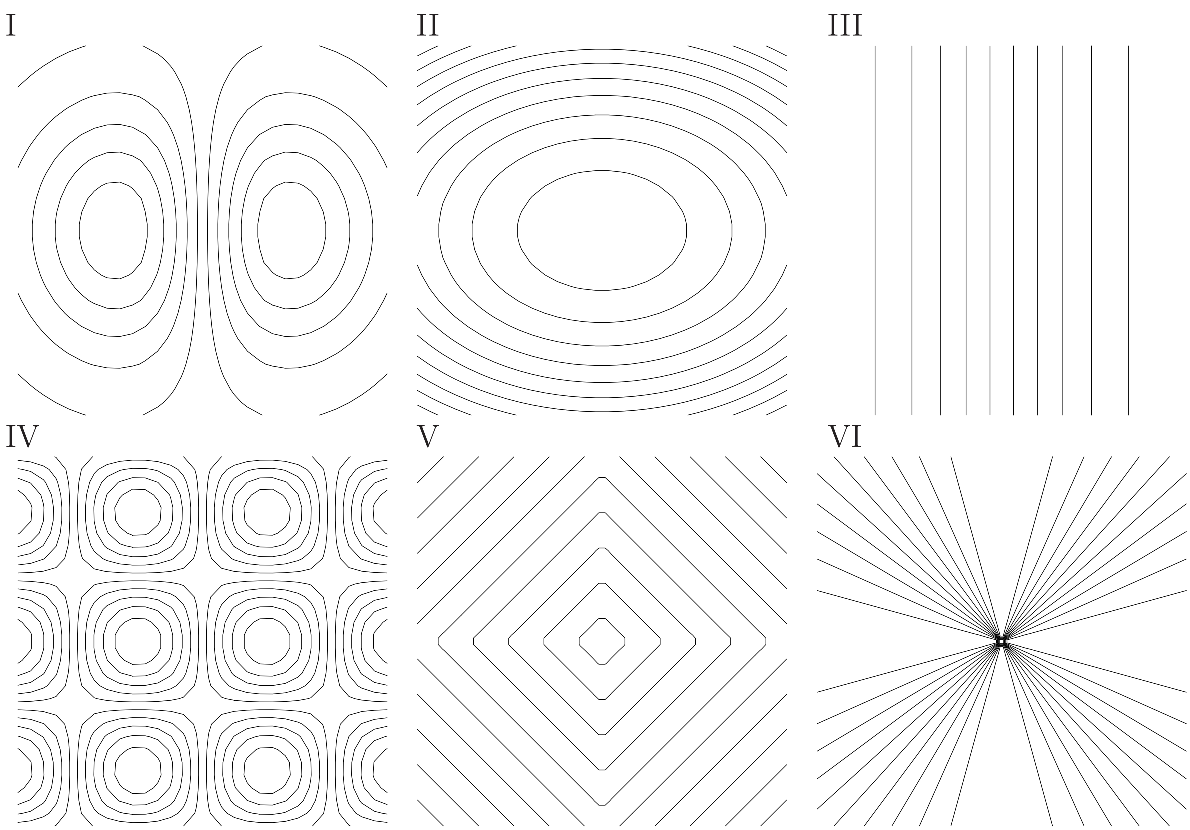
Enter I,II,III,IV,V,VI here	Equation or Parameterization
	$\vec{r}(u, v) = ((1 + \sin(u))\cos(v), (1 + \sin(u))\sin(v), \cos(u))$
	$\vec{r}(u, v) = (v, v - u, u + v)$
	$\vec{r}(u, v) = (u^2, vu, v)$
	$x^2 - y^2 + z^2 - 1 = 0$
	$\vec{r}(u, v) = (\cos(u)\sin(v), \cos(v), \sin(u)\sin(v))$

Solution:

Enter I,II,III,IV,V,VI here	Equation or Parameterization
II	$\vec{r}(u,v) = ((1+\sin(u))\cos(v), (1+\sin(u))\sin(v), \cos(u))$
IV	$\vec{r}(u,v) = (v, v-u, u+v)$
I	$\vec{r}(u,v) = (u^2, vu, v)$
III	$x^2 - y^2 + z^2 - 1 = 0$
V	$\vec{r}(u,v) = (\cos(u)\sin(v), \cos(v), \sin(u)\sin(v))$

Problem 2b) (5 points)

Match the contour maps with the corresponding functions $f(x,y)$ of two variables. No justifications are needed.



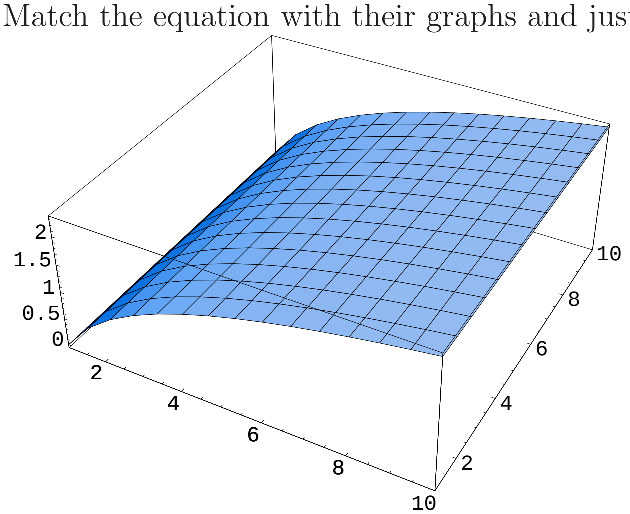
Enter I,II,III,IV,V or VI here	Function $f(x, y)$
	$f(x, y) = \sin(x)$
	$f(x, y) = x^2 + 2y^2$
	$f(x, y) = \|x\| + \|y\|$
	$f(x, y) = \sin(x)\cos(y)$
	$f(x, y) = xe^{-x^2-y^2}$
	$f(x, y) = x^2/(x^2 + y^2)$

Solution:

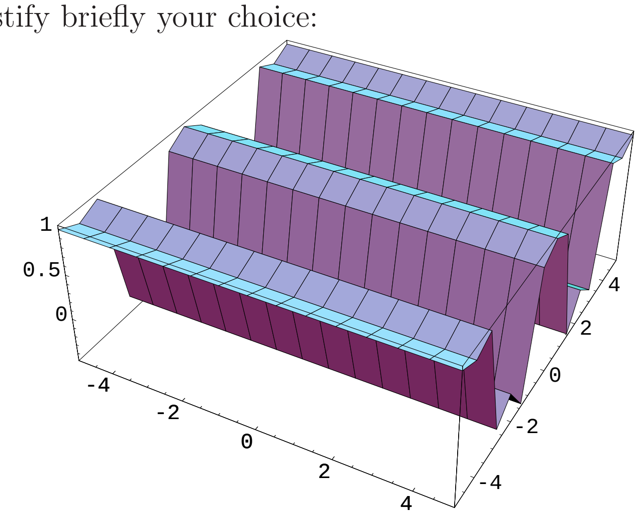
Enter I,II,III,IV,V or VI here	Function $f(x, y)$
III	$f(x, y) = \sin(x)$
II	$f(x, y) = x^2 + 2y^2$
V	$f(x, y) = \|x\| + \|y\|$
I	$f(x, y) = xe^{-x^2-y^2}$
VI	$f(x, y) = x^2/(x^2 + y^2)$

Problem 3) (10 points)

Match the equation with their graphs and justify briefly your choice:



I



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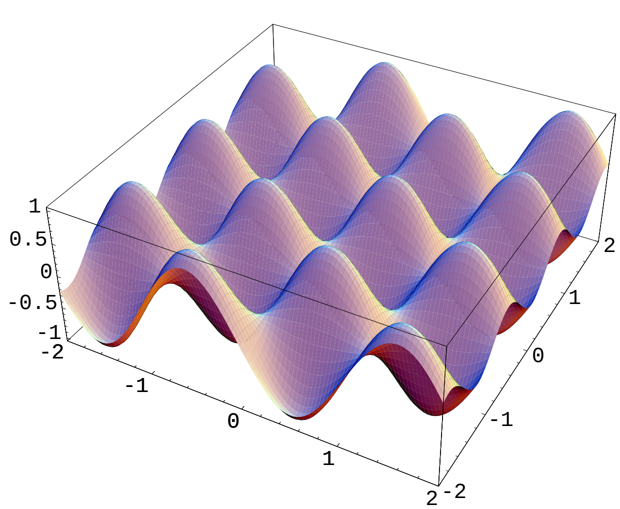

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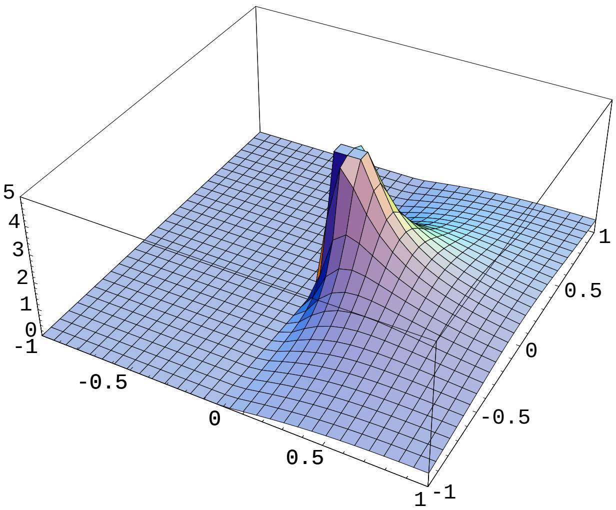


IV

Enter I,II,III,IV here	Equation	Short Justification
	$z = \sin(3x)\cos(5y)$	
	$z = \cos(y^2)$	
	$z = \log(x)$	
	$z = x/(x^2 + y^2)$	

Solution:

Enter I,II,III,IV here	Equation	Short Justification
III	$z = \sin(3x)\cos(5y)$	two traces show waves
II	$z = \cos(y^2)$	no x dependence, periodic in y
I	$z = \log(x)$	no y dependence, monotone in x
IV	$z = x/(x^2 + y^2)$	singular at (x,y)=(0,0)

Problem 4) Distances (10 points)

Let L be the line

$$x = 1 + 2t, y = -3t, z = t$$

and let S be the plane $x + y + z = 2$.

a) Verify that L and S have no intersections.

b) Compute the distance between the line L and plane S.

Hint. Just take any point P on the line and compute the distance from the line to the plane.

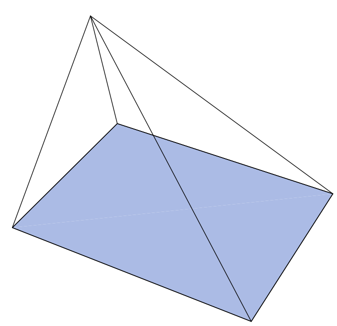
Solution:
The vector $v = (2, -3, 1)$ is in the line. It is is normal to the normal vector $(1, 1, 1)$ of the plane.
The point $P = (1, 0, 0)$ is on the line. The point $Q = (1, 1, 0)$ is on the plane. The distance is the scalar projection of PQ onto the normal vector $(1, 1, 1)$ which is $(0, 1, 0) \cdot (1, 1, 1)/\sqrt{3} = 1/\sqrt{3}$

Problem 5) (10 points)

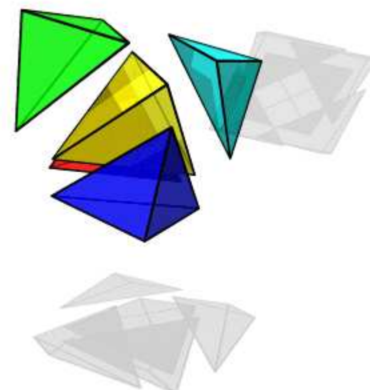
a) (5 points) Find the area of the parallelogram $PQSR$ with corners

$$P = (0, 0, 0), Q = (1, 1, 1), R = (1, 1, 0), S = (2, 2, 1) .$$

b) (5 points) Find the volume of the pyramid which has as the base the parallelogram $PQRS$ and has a fifth vertex at $T = (3, 4, 3)$.



Solution:
a) The area is $|(Q-P)\times(R-P)| = \sqrt{2}$.
b) The distance $h = |w \cdot (u \times v)|/|w|$ of T to the plane containing the points P,Q,S,R and use the volume formula $V = Ah/3$, where $A = |u \times v|$ is the area of the parallelogram. The volume of the pyramid is $\boxed{1/3}$.



Remark. The parallelepiped spanned by $\vec{u} = (1,1,1), \vec{v} = (2,2,1)$ and $\vec{w} = (3,4,3)$ has volume 1. The pyramid has volume 1/3 of this volume because one can chop the pyramid into two tetraedra of the same volume and each tetrahedron has volume 1/6. One can take 4 tetrahedra spanned by u,v,w as well as one of double volume spanned by $(u+v, u+w, v+w)$ to build the parallelepiped. The volume of the pyramid is therefore $\boxed{1/3}$.
The following picture shows that a tetrahedron spanned by vectors u,v,w has a volume 1/6'th of the volume of the parallelepiped:

Problem 6) (10 points)

Find the distance between the two lines

$$\vec{r}_1(t) = \langle t, 2t, -t \rangle$$

and

$$\vec{r}_2(t) = \langle 1+t, t, t \rangle \ .$$

Solution:
The point $P = (0,0,0)$ is on the first line. The point $Q = (1,0,0)$ on the second line. The vector $\vec{v} = \langle 1,2,-1 \rangle$ in the first line and $\vec{w} = \langle 1,1,1 \rangle$ in the second line. We have $\vec{n} = \langle 3,-2,-1 \rangle$. Now, the distance is $3/\sqrt{14}$. $(Q-P)\cdot\vec{n}/|\vec{n}| = \langle 1,0,0 \rangle \cdot \langle 3,-2,-1 \rangle/|n| = 3/\sqrt{14}$.

Problem 7) (10 points)

Given the vectors $v = (1,1,0)$ and $w = (0,0,1)$ and the point $P = (2,4,-2)$. Let Σ be the plane which goes through the origin and contains the vectors $\vec{v}$ and $\vec{w}$.

a) Determine the distance from P to the origin.

Solution:
$\sqrt{4+16+4} = \sqrt{24} = 2\sqrt{6}$.

b) Determine the distance from P to the plane Σ.

Solution:
$\Sigma : x - y = 0$, $n = (1, -1, 0)$. $Q = (0, 0, 0)$ is a point on the plane. $\vec{PQ} \cdot n/|n| = (2, 4, -2) \cdot (1, -1, 0)/\sqrt{2} = 2/\sqrt{2} = \sqrt{2}$

Problem 8) (10 points)

In this problem, it is enough to describe the surface with words.

a) (3) Identify the surface whose equation is given in spherical coordinates as $\phi = \pi/6$.

b) (3) Identify the surface whose equation is given in spherical coordinates as $\theta = \pi/2$.

c) (2) Identify the surface, whose equation is given in cylindrical coordinates by $z^2 = r$.

d) (2) Identify the surface, whose equation is given in cylindrical coordinates as $r\cos(\theta) = 1$

Solution:
a) A **half cone**. $x^2 + y^2 = z^2$ with $z \geq 0$.
b) A **half plane** contained in the plane $x = 0$ and containing the positive y axis.
c) This surface appeared in the homework. It is a "concave cone-type" surface of revolution. The trace can be obtained by drawing $z = \pm\sqrt{r}$. If you spin this graph wround the z axis, you obtain the surface.
d) This is the **plane** $x = 1$. Just note that $r\cos(\theta) = x$ in cylindrical coordinates.

Problem 9) (10 points)

a) (3 points) Let S be the surface $g(x, y, z) = x^2 - z - y^2 = 1$. Find a parametrization

$$\vec{r}(u, v) = (x(u, v), y(u, v), z(u, v))$$

of this surface.

b) (3 points) Write down the parametrization

$$\vec{r}(u, v) = (x(u, v), y(u, v), z(u, v))$$

of the part of the unit sphere $x^2 + y^2 + z^2 = 1$ which satisfies $z \geq \sqrt{3}/2$ and also indicate the domain R of the parametrization.

c) (4 points) Let S be the surface given in cylindrical coordinates as $r = 2 + \sin(z)$. Find a parameterization

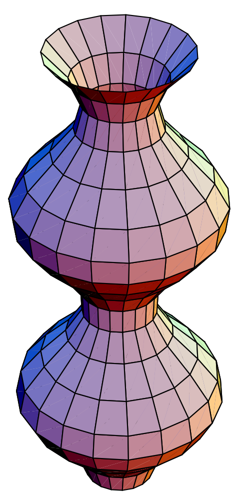
$$\vec{r}(u, v) = (x(u, v), y(u, v), z(u, v))$$

of the surface.

Solution:

a) This is a graph. We can therefore write $\vec{r}(u, v) = (u, v, u^2 - v^2 - 1)$.

b) $\vec{r}(u, v) = (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$ with $0 \leq u \leq 2\pi$ and $0 \leq v \leq \pi/6$. c) The distance to the z-axis is $2 + \sin(z)$. We take the rotation angle θ as a second parameter. Therefore $\vec{r}(\theta, z) = ((2 + \sin(z))\cos(\theta), (2 + \sin(z))\sin(\theta), z)$.



Problem 10) (10 points)

Remember that a parameterization of a surface describes the points (x, y, z) of the surface in the form $\vec{r}(u, v) = (x, y, z) = (x(u, v), y(u, v), z(u, v))$. What surfaces do the following parameterizations represent? Find in each case an implicit equation of the form $g(x, y, z) = c$ which is equivalent.

a) (3) $\vec{r}(u, v) = (\cos(u), \sin(u), v)$

b) (3) $\vec{r}(u, v) = (u + v, v - u, u + 2v)$

c) (2) $\vec{r}(u, v) = (v\cos(u), v\sin(u), v)$

d) (2) $\vec{r}(u, v) = (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$.

Solution:
a) This is a surface of revolution. It is a cylinder $x^2 + y^2 = 1$. The general form of a surface of revolutio nis $(g(v)cos(u), g(v)sin(u), v)$. Here, $g(v)$ the distance to the z-axes is constant equal to 1.
b) This is a plane containing the vectors $(1, -1, 1)$ and $(1, 1, 2)$. To get the equation of the plane, take the cross product. Because the plane passes through the origin, we have $-3x - y + 2z = 0$.
c) This is a cone $x^2 + y^2 = z^2$. Also, here, we have a surface of revolution. The function $g(v)$ satisfies $g(v) = v$.
d) This is the unit sphere: $x^2 + y^2 + z^2 = 1$.

Problem 11) (10 points)

Find an equation for the plane that passes through the origin and whose normal vector is parallel to the line of intersection of the planes $2x + y + z = 4$ and $x + 3y + z = 2$.

Solution:
The line of intersection is parallel to the crossed product of $\vec{v} = \langle 2, 1, 1\rangle$ and $\vec{w} = \langle 1, 3, 1\rangle$ which is $\langle -2, -1, 5\rangle$. This vector is perpendicular to the plane we are looking for. So, we know that the equation is $-2x - y + 5z = d$. The constant d can be obtained by plugging in a point on the plane. Because $(0, 0, 0)$ is on the plane by assumption, we know that $d = 0$. Therefore, the equation of the plane is $\boxed{-2x - y + 5z = 0}$.