7/27/2006 SECOND HOURLY PRACTICE   Maths21a, O. Knill, Summer 2006

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1)	(10 points)

T **F** The speed of the curve $\vec{r}(t) = (\cos(t), \sin(t), 3t)$ is $(-\sin(t), \cos(t), 3)$.

Solution:
This is the velocity. The speed is the length of this vector and would be $\sqrt{10}$.

T F Every smooth function of three variables $f(x, y, z)$ satisfies the partial differential equation $f_{xyz} + f_{yzx} = 2f_{zxy}$.

Solution:
By Clairot's theorem.

T **F** If $f_x(x, y) = f_y(x, y)$ for all x, y, then $f(x, y)$ is a constant.

Solution:
The transport equation $f_x = f_y$ is a PDE with solutions like for example $f(x, y) = x + y$. Any function which stays invariant by replacing x with y is a solution: like $f(x, y) = \sin(xy) + x^5 y^5$.

T **F** $(1, 1)$ is a local maximum of the function $f(x, y) = x^2 y - x + \cos(y)$.

Solution:
$(1, 1)$ is not even a critical point.

T **F** If f is a smooth function of two variables, then the number of critical points of f inside the unit disc is finite.

Solution:
Take $f(x, y) = x^2$ for example. Every point on the y axis $\{x = 0\}$ is a critical point.

T **F** The value of the function $f(x, y) = \sin(-x + 2y)$ at $(0.001, -0.002)$ can by linear approximation be estimated as -0.003.

Solution:
The correct approximation would be $f(0, 0) + 0.001(-1) - 0.002(2) = -0.005$.

T | F — If $(1,1)$ is a critical point for the function $f(x,y)$ then $(1,1)$ is also a critical point for the function $g(x,y) = f(x^2,y^2)$.

Solution:
If $\nabla f(1,1) = (f_x(1,1), f_y(1,1)) = (0,0)$ then also $\nabla g(1,1) = (f_x(1,1)2x, f_y(1,1)2y) = (0,0)$. Note that the statement would not be true, if we would replace $(1,1)$ say with $(2,2)$ (as in the practice exam).

T | F — If the velocity vector $\vec{r}\,'(t)$ of the planar curve $\vec{r}(t)$ is orthogonal to the vector $\vec{r}(t)$ for all times t, then the curve is a circle.

Solution:
$d/dt(\vec{r}(t) \cdot \vec{r}(t)) = \vec{r}(t) \cdot \vec{r}'(t)$.

T | F — The curvature of a circle of radius 10 is $1/10$.

Solution:
Yes, the curvature of a circle of radius r is $1/10$.

T | **F** — The arc length of a curve is given by the formula $\int_a^b \vec{r}'(t)\, dt$.

Solution:
Take absolute values.

T | F — The gradient of $f(x,y)$ is normal to the level curves of f.

Solution:
This is a basic and important fact.

T | **F** — If (x_0,y_0) is a maximum of $f(x,y)$ under the constraint $g(x,y) = g(x_0,y_0)$, then (x_0,y_0) is a maximum of $g(x,y)$ under the constraint $f(x,y) = f(x_0,y_0)$.

Solution:
Assume you have a situation f, g, where this is true and where the constraint is $g = 0$, produce a new situation $f, h = -g$, where the first statement is still true but where the extrema of h under the constraint of f is a minimum.

T	F

If $\vec{u}$ is a unit vector tangent at (x, y, z) to the level surface of $f(x, y, z)$ then the directional derivative satisfies $D_u f(x, y, z) = 0$.

Solution:
The directional derivative measures the rate of change of f in the direction of u. On a level surface, in the direction of the surface, the function does not change (because f is constant by definition on the surface).

T	F

If $\vec{r}(t) = (x(t), y(t))$ and $x(t), y(t)$ are polynomials, then the tangent line is defined at all points.

Solution:
Take the example $r(t) = (t^2, t^3)$. At $t = 0$, we have a cusp and the gradient is zero. We do not have a tangent line there.

T	F

The function $u(x, t) = x^2/2 + t$ satisfies the heat equation $u_t = u_{xx}$.

Solution:
Just differentiate.

T	F

The vector $\vec{r}_u(u, v)$ is tangent to the surface parameterized by $\vec{r}(u, v) = (x(u, v), y(u, v), z(u, v))$.

Solution:
The vector $\vec{r}_u$ is tangent to a grid curve and so tangent to the surface.

T	F

Clairots theorem implies $f_{xyx} = f_{yxy}$

Solution:
We have $f_{xy} = f_{yx}$ but in the formula, we have to the left two x derivatives and to the right two y derivatives.

T **F** The second derivative test allows to check whether an extremum found with the Lagrange multiplyer method is a maximum.

Solution:
No, the second derivative test applies for function $f(x, y)$ without constraint.

T F If $(0, 0)$ is a critical point of $f(x, y)$ and the discriminant D is zero but $f_{xx}(0, 0) > 0$ then $(0, 0)$ can not be a local maximum.

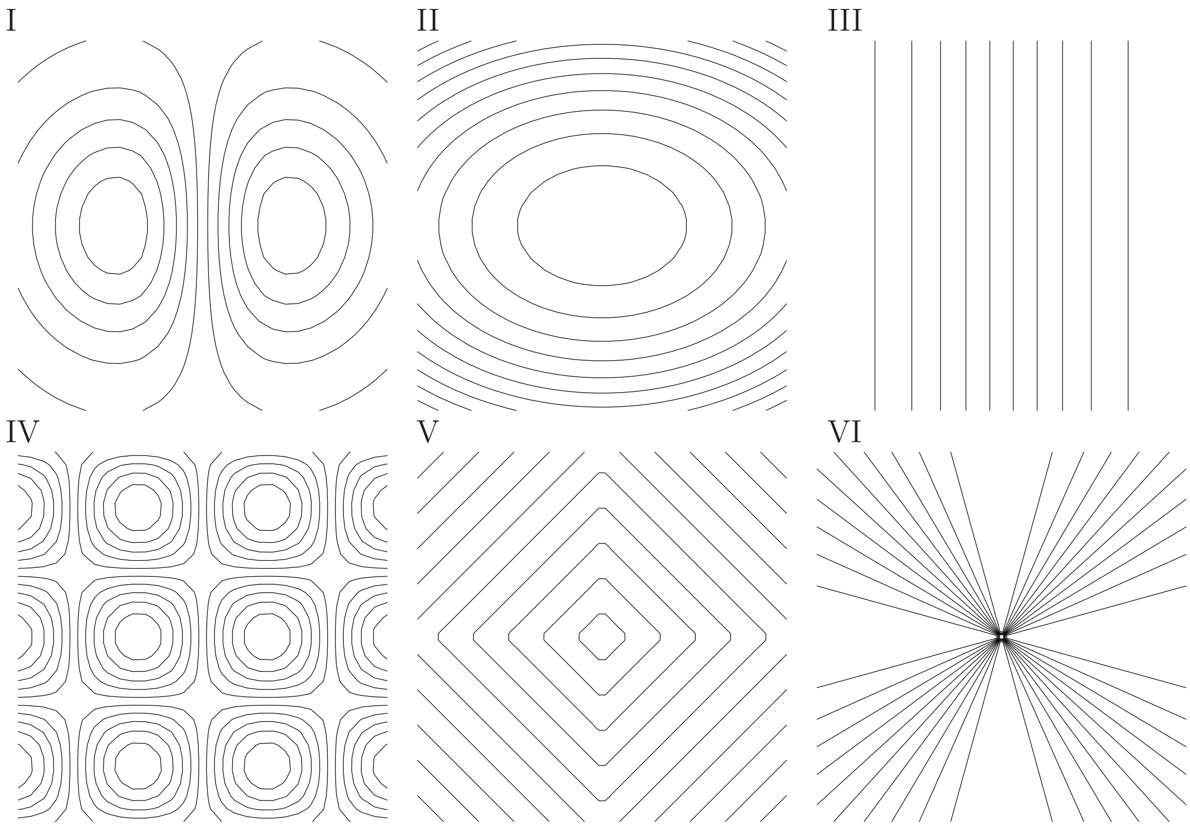
Solution:
If $f_{xx}(0, 0) > 0$ then on the x-axis the function $g(x) = f(x, 0)$ has a local minimum. This means that there are points close to $(0, 0)$ where the value of f is larger.

T F Let (x_0, y_0) be a saddle point of $f(x, y)$. For any unit vector $\vec{u}$, there are points arbitrarily close to (x_0, y_0) for which ∇f is parallel to $\vec{u}$.

Solution:
Just look at the level curves near a saddle point. The gradient vectors are orthogonal to the level curves which are hyperbola. You see that they point in any direction except 4 directions. To see this better, take a pen and draw a circle around the saddle point between two of your knuckles on your fist. At each point of the circle, now draw the direction of steepest increase (this is the gradient direction).

Problem 2) (10 points)

Match the contour maps with the corresponding functions $f(x,y)$ of two variables. Note that one of the contour maps is not represented by a formula. No justifications are needed.

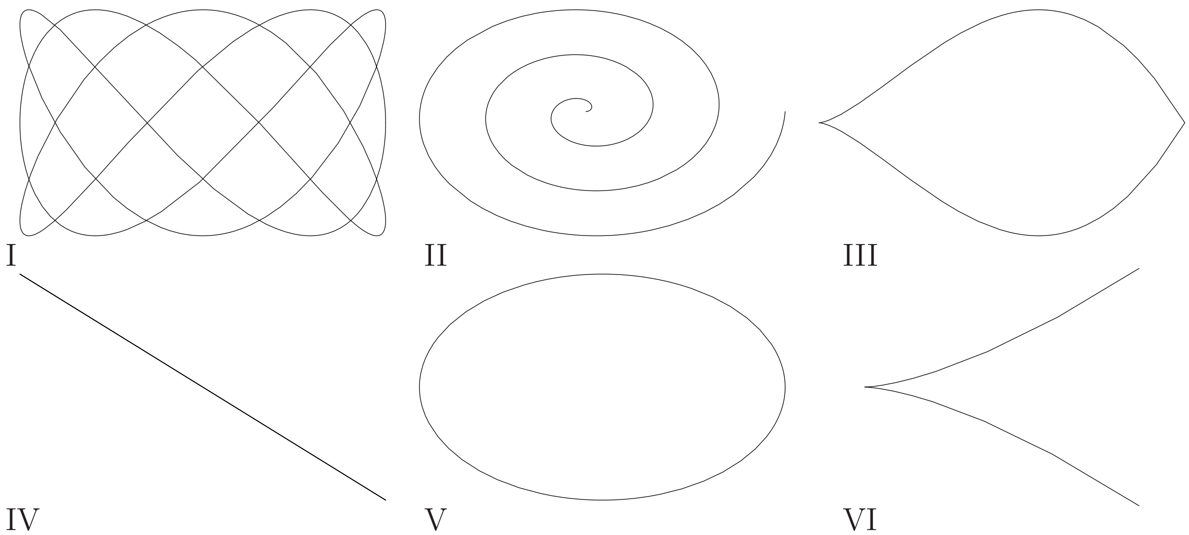


Enter I,II,III,IV,V or VI here	Function $f(x,y)$
	$f(x,y) = \sin(x)$
	$f(x,y) = x^2 + 2y^2$
	$f(x,y) = \|x\| + \|y\|$
	$f(x,y) = xe^{-x^2-y^2}$
	$f(x,y) = x^2/(x^2+y^2)$

Solution:

Enter I,II,III,IV,V or VI here	Function $f(x,y)$
III	$f(x,y) = \sin(x)$
II	$f(x,y) = x^2 + 2y^2$
V	$f(x,y) = \|x\| + \|y\|$
I	$f(x,y) = xe^{-x^2-y^2}$
VI	$f(x,y) = x^2/(x^2+y^2)$

Match the Parametrizations with the curves. No justifications are needed.



Enter I,II,..., until VI here	Equation
	$\vec{r}(t) = (\cos(t), \sin(t))$
	$\vec{r}(t) = (\cos(3t), \sin(5t))$
	$\vec{r}(t) = (t\cos(3t), t\sin(3t))$
	$\vec{r}(t) = (t^2, t^3 - t^5)$
	$\vec{r}(t) = (\cos(t)^2, \sin(t)^2)$
	$\vec{r}(t) = (t^2, t^3)$

Solution:
V I II III IV VI

Problem 3) (10 points)

a) Use the technique of linear approximation to estimate $f(\log(2) + 0.001, 0.006)$ for $f(x, y) = e^{2x-y}$. (Here, log means the natural logarithm).
b) Find the equation $ax + by = d$ for the tangent line which goes through the point $(\log(2), 0)$.

Solution:
a) $L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$
$f(x_0,y_0) = e^{2\log 2} = 4$
$f_x(x_0,y_0) = 8$
$f_y(x_0,y_0) = -4$
$L(x,y) = 4 + 0.001 \cdot 8 - 4 \cdot 0.006 = \boxed{3.984}$.
b) We have $a = 8$ and $b = -4$ and get $d = 8\log(2)$ so that the line has the equation $\boxed{8x - 4y = 8\log(2)}$.

Problem 4)	(10 points)

Find a point on the surface $g(x,y,z) = \frac{1}{x} + \frac{1}{y} + \frac{8}{z} = 1$ for which the distance to the origin is a local minimum.

Solution:
This is a Lagrange problem. One wants to minimize $f(x,y,z) = x^2 + y^2 + z^2$ under the constraint $g(x,y,z) = 1$. The Lagrange equations are

$$\begin{aligned} \frac{-1}{x^2} &= 2\lambda x \\ \frac{-1}{y^2} &= 2\lambda y \\ \frac{-8}{z^2} &= 2\lambda z \\ \frac{1}{x} + \frac{1}{y} + \frac{8}{z} &= 1 \end{aligned}$$

The first two equations show $x = y$, the first and third equations show $8/z^3 = 1/x^3$ or $z = 2x$. Plugging this into the last equation gives $2/x + 8/(2x) = 1$ or $x = 6, y = 6, z = 12$. $\boxed{(x,y,z) = (6,6,12)}$.
The global picture is interesting: consider the points $(x,y,z) = (1, -1/n, 8/n)$, where n is a large integer, One can check that these points ly on the surface $g(x,y,z) = 1$. Their distance to the origin decreases to 1 if n goes to infinity. So the point $(6,6,12)$, while a local minimum is not a global minimum.

Problem 5)	(10 points)

Find all extrema of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$ on the plane and characterize them. Do you find a absolute maximum or absolute minimum amoung them?

Solution:
The critical points satisfy $\nabla f(x,y) = (0,0)$ or $(3x^2-3, 3y^2-12) = (0,0)$. There are 4 critical points $(x,y) = (\pm 1, \pm 2)$. The discriminant is $D = f_{xx}f_{yy} - f_{xy}^2 = 36xy$ and $f_{xx} = 6x$.

point	D	f_{xx}	classification	value
(-1,-2)	72	-6	maximum	38
(-1, 2)	-72	-6	saddle	6
(1, -2)	-72	6	saddle	34
(1, 2)	72	6	minimum	2

Note that there are no global (= absolute) maxima nor global minima because the function takes arbitrarily large and small values. For $y = 0$ the function is $g(x) = f(x,0) = x^3 - 3x + 20$ which satisfies $\lim_{x\to\pm\infty} g(x) = \pm\infty$.

Problem 6) (10 points)

A skydiver feels the acceleration $\vec{r}\,''(t) = (0,1,-10)$ which is the sum of a gravitational pull $(0,0,-10)$ and a wind force $(0,1,0)$. Assume the skydiver jumps at $t = 0$ from the plane at the position $\vec{r}(0) = (10,10,1000)$ and with velocity $\vec{r}\,'(0) = (100,0,0)$. Find the position of the skydiver at time $t = 10$.

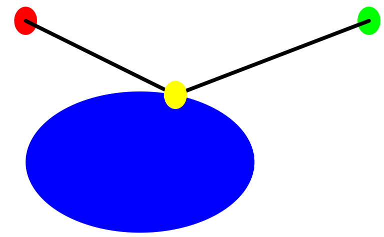
Solution:
$\vec{r}(t) = \vec{r}(0) + t\vec{r}\,'(0) + t^2/2\vec{r}\,''(0) = (10,10,1000) + t(100,0,0) + t^2/2(0,1,-10) = (1010,60,500)$.

Problem 7) (10 points)

Find the tangent plane to the surface $f(x,y,z) = x^3y - xy^2 + 3z = 6$ at the point $(1,1,2)$.

Solution:
The gradient of f is $\nabla f(x,y,z) = (3x^2y - y^2, x^3 - 2xy, 3)$ which is at the point $(1,1,2)$ equal to $\nabla f(1,1,2) = (2,-1,3)$. The plane has therefore the equation $2x - y + 3z = d$. The constant d can be obtained from plugging in the coordinates of the point $d = 7$. The final answer is $2x - y + 3z = 7$.

Problem 8) (10 points)



You find yourself in the desert at the point $A = (a, 1)$, completely dehydrated and almost dead. You want to reach the point $B = (b, 1)$ as fast as possible but you can not reach it without water. There is an lake inside the ellipsoid $g(x, y) = x^2 + 2y^2 = 1$. The amount of "effort" you need to go from a point (x, y) to a point (u, v) is assumed to be $(x - u)^2 + (y - v)^2$ (this is justified by the fact that if you walk for a long time, you walk less and less efficiently so that walking twice as long will take you 4 times as much effort). Find the path of least effort which connects A with $X = (x, y)$ and then with B.

a) Which function $f(x, y)$ do you extremize? The parameters a, b are constants.
b) Write down the Lagrange equations.
c) Solve the Lagrange equations in the case $a = -1, b = 1$.

Solution:
a) We have to extremize $f(x, y) = (x - a)^2 + (y - 1)^2 + (x - b)^2 + (y - 1)^2$ under the constraint $x^2 + 2y^2 = 1$.
b) The Lagrange equations are

$$\begin{aligned} 2(x - a) + 2(x - b) &= 2\lambda x \\ 4(y - 1) &= 4\lambda y \\ x^2 + 2y^2 &= 1 \end{aligned}$$

c) In the case $a = -1, b = 1$ we have extremal solutions $(0, 1/\sqrt{2})$ and $(0, -1/\sqrt{2})$. The first one is the minimum, the second the maximum.

Problem 9) (10 points)

Let $\vec{r}(t)$ be the space curve $\vec{r}(t) = (t^2, \sin(3\pi t), \cos(5\pi t))$.

a) Calculate the velocity, the acceleration and the speed of $\vec{r}(t)$ at time $t = 1$.
b) Write down the integral for the arc length of the curve from $t = 1$ to $t = 10$ as an integral. You don't have to evaluate the integral.
c) The curve $t \mapsto \vec{r}(t) = (t^3, 1 - t, 1 - t^3)$ lies in a plane. What is the equation of this plane?

Solution:

a) $\vec{v}(t) = \vec{r}'(t) = (2t, 3\pi\cos(3\pi t), -5\pi\sin(5\pi t))$.
$\vec{v}(1) = (2, -3\pi, 0)$
$\vec{a}(t) = \vec{r}''(t) = (2, -9\pi^2\sin(3\pi t), -25\pi^2\cos(5\pi t))$.
$\vec{a}(1) = (2, 0, 25\pi^2)$. $|v(1)| = \sqrt{4 + 9\pi^2}$.
b) $\int_1^{10} \sqrt{4t^2 + 9\pi^2\cos^2(3\pi t) + 25\pi^2\sin^2(5\pi t)}$.
c) $t = 0 : P = (0, 1, 1), t = 1 : Q = (1, 0, 0), t = 2 : R = (8, -1, -7)$ are points on the Plane. $\vec{PQ} = (1, -1, -1)$, $\vec{PR} = (8, -2, -8)$. Their cross product is $(6, 0, 6)$. The plane is $x + z = 1$.