

## VECTORS, DOT PRODUCT

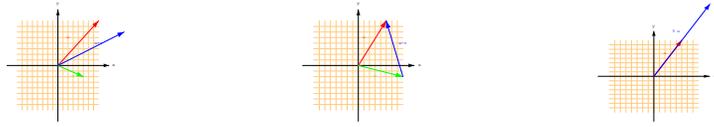
O. Knill, Maths21a

**VECTORS.** Two points  $P = (x_1, y_1)$ ,  $Q = (x_2, y_2)$  in the plane define a **vector**  $\vec{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$ . It points from  $P$  to  $Q$  and we can write  $P + \vec{v} = Q$ .

**COMPONENTS.** Points  $P$  in space are in one to one correspondence to vectors pointing from 0 to  $P$ . The numbers  $\vec{v}_i$  in a vector  $\vec{v} = \langle v_1, v_2 \rangle$  are also called **components** or of the vector.

**REMARKS:** vectors can be drawn **everywhere** in the plane. If a vector starts at the origin  $O$ , then the vector  $\vec{v} = \langle v_1, v_2 \rangle$  points to the point  $\langle v_1, v_2 \rangle$ . That's is why one can identify points  $P = (a, b)$  with vectors  $\vec{v} = \langle a, b \rangle$ . Two vectors which can be translated into each other are considered **equal**. They have the same components.

### ADDITION, SUBTRACTION AND SCALAR MULTIPLICATION.



$$\vec{u} + \vec{v} = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle = \langle u_1 + v_1, u_2 + v_2 \rangle$$

$$\vec{u} - \vec{v} = \langle u_1, u_2 \rangle - \langle v_1, v_2 \rangle = \langle u_1 - v_1, u_2 - v_2 \rangle$$

$$\lambda \vec{u} = \lambda \langle u_1, u_2 \rangle = \langle \lambda u_1, \lambda u_2 \rangle$$

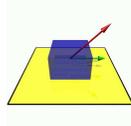
**BASIS VECTORS.** The vectors  $\vec{i} = \langle 1, 0 \rangle$ ,  $\vec{j} = \langle 0, 1 \rangle$  are called **standard basis vectors** in the plane. In space, one has the basis vectors  $\vec{i} = \langle 1, 0, 0 \rangle$ ,  $\vec{j} = \langle 0, 1, 0 \rangle$ ,  $\vec{k} = \langle 0, 0, 1 \rangle$ . Every vector  $\vec{v} = \langle v_1, v_2 \rangle$  in the plane can be written as  $\vec{v} = v_1 \vec{i} + v_2 \vec{j}$ . Every vector  $\vec{v} = \langle v_1, v_2, v_3 \rangle$  in space can be written as  $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$ .

### WHERE DO VECTORS OCCUR? Here are some examples:

**Velocity:** vectors appear in mechanics. For example, if  $(f(t), g(t))$  is a point in the plane which depends on time  $t$ , then  $\vec{v} = \langle f'(t), g'(t) \rangle$  is the **velocity vector** at the point  $(f(t), g(t))$ . We will look at this in more detail later in the course.



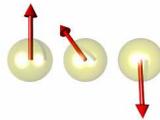
**Forces:** Some problems in statics involve the determination of a forces acting on objects. Forces are represented as vectors



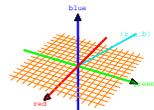
**Fields:** electromagnetic or gravitational fields or velocity fields in fluids are described with vectors.



**Qbits:** in quantum computation, rather than working with bits, one deals with **qbits**, which are vectors.



**Color** can be written as a vector  $\vec{v} = \langle r, g, b \rangle$ , where  $r$  is **red**,  $g$  is **green** and  $b$  is **blue**. An other coordinate system is  $\vec{v} = \langle c, m, y \rangle = (1 - r, 1 - g, 1 - b)$ , where  $c$  is **cyan**,  $m$  is **magenta** and  $y$  is **yellow**.



**SVG.** Scalable Vector Graphics is an emerging standard for the web for describing two-dimensional graphics in XML.



**VECTOR OPERATIONS:** The addition and scalar multiplication of vectors satisfy "obvious" properties. There is no need to memorize them. We write  $*$  here for multiplication with a scalar but usually, the multiplication sign is left out.

$\vec{u} + \vec{v} = \vec{v} + \vec{u}$	commutativity
$\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$	additive associativity
$\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$	null vector
$r * (s * \vec{v}) = (r * s) * \vec{v}$	scalar associativity
$(r + s)\vec{v} = \vec{v}(r + s)$	distributivity in scalar
$r(\vec{v} + \vec{w}) = r\vec{v} + r\vec{w}$	distributivity in vector
$1 * \vec{v} = \vec{v}$	the one element

**LENGTH.** The length  $|\vec{v}|$  of  $\vec{v}$  is the distance from the beginning to the end of the vector.

**EXAMPLES.** 1) If  $\vec{v} = \langle 3, 4 \rangle$ , then  $|\vec{v}| = \sqrt{25} = 5$ . 2)  $|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$ ,  $|\vec{0}| = 0$ .

**UNIT VECTOR.** A vector of length 1 is called a **unit vector**. If  $\vec{v} \neq \vec{0}$ , then  $\vec{v}/|\vec{v}|$  is a unit vector.

**EXAMPLE:** If  $\vec{v} = \langle 3, 4 \rangle$ , then  $\vec{v} = (3/5, 4/5)$  is a unit vector,  $\vec{i}, \vec{j}, \vec{k}$  are unit vectors.

**PARALLEL VECTORS.** Two vectors  $\vec{v}$  and  $\vec{w}$  are called **parallel**, if  $\vec{v} = r\vec{w}$  with some constant  $r$ .

**DOT PRODUCT.** The **dot product** of two vectors  $\vec{v} = \langle v_1, v_2, v_3 \rangle$  and  $\vec{w} = \langle w_1, w_2, w_3 \rangle$  is defined as

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Remark: in other sciences different notations are used:  $\vec{v} \cdot \vec{w} = (\vec{v}, \vec{w})$  (mathematics)  $\langle \vec{v} | \vec{w} \rangle$  (quantum mechanics)  $v_i w^i$  (Einstein notation)  $g_{ij} v^i w^j$  (general relativity). The dot product is also called **scalar product**, or **inner product**.

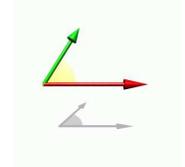
**LENGTH.** Using the dot product one can express the length of  $\vec{v}$  as  $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$ .

**CHALLENGE.** Express the dot product in terms of the length alone.

**SOLUTION:**  $(\vec{v} + \vec{w}, \vec{v} + \vec{w}) = (\vec{v}, \vec{v}) + (\vec{w}, \vec{w}) + 2(\vec{v}, \vec{w})$  can be solved for  $(\vec{v}, \vec{w})$ .

**ANGLE.** Because  $|\vec{v} - \vec{w}|^2 = (\vec{v} - \vec{w}, \vec{v} - \vec{w}) = |\vec{v}|^2 + |\vec{w}|^2 - 2(\vec{v}, \vec{w})$ , which is by the **cos-theorem** equal to  $|\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}||\vec{w}|\cos(\alpha)$ , with the angle  $\alpha$  between  $\vec{v}, \vec{w}$ , we obtain the **dot product formula**

$$\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$$



**CAUCHY-SCHWARZ INEQUALITY:**  $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$  follows from that formula because  $|\cos(\alpha)| \leq 1$ .

**TRIANGLE INEQUALITY:**  $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$  follows from  $|\vec{u} + \vec{v}|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} + 2\vec{u} \cdot \vec{v} \leq \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} + 2|\vec{u}| \cdot |\vec{v}| = (|\vec{u}| + |\vec{v}|)^2$ .

**FINDING ANGLES BETWEEN VECTORS.** Find the angle between the vectors  $\langle 1, 4, 3 \rangle$  and  $\langle -1, 2, 3 \rangle$ .

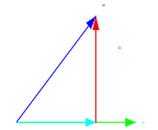
**ANSWER:**  $\cos(\alpha) = 16/(\sqrt{26}\sqrt{14}) \sim 0.839$ . So that  $\alpha = \arccos(0.839) \sim 33^\circ$ .

**ORTHOGONAL VECTORS.** Two vectors are called **orthogonal** (= **perpendicular**) if  $\vec{v} \cdot \vec{w} = 0$ . The zero vector  $\vec{0}$  is orthogonal to any vector. **EXAMPLE:**  $\vec{v} = \langle 2, 3 \rangle$  is orthogonal to  $\vec{w} = \langle -3, 2 \rangle$ .

**PYTHAGORAS.** If  $\vec{v}$  and  $\vec{w}$  are orthogonal, then  $|\vec{v} - \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2$ . Proof:  $(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} + 2\vec{v} \cdot \vec{w} = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w}$ .

**PROJECTION.** The vector  $\vec{a} = \text{proj}_{\vec{w}}(\vec{v}) = \vec{w}(\vec{v} \cdot \vec{w} / |\vec{w}|^2)$  is called the **projection** of  $\vec{v}$  onto  $\vec{w}$ .

The **scalar projection** is defined as  $\text{comp}_{\vec{w}}(\vec{v}) = (\vec{v} \cdot \vec{w}) / |\vec{w}|$ . Its absolute value is the length of the projection of  $\vec{v}$  onto  $\vec{w}$ . The vector  $\vec{b} = \vec{v} - \vec{a}$  is called the **component** of  $\vec{v}$  orthogonal to the  $\vec{w}$ -direction.



**EXAMPLE.**  $\vec{v} = \langle 0, -1, 1 \rangle$ ,  $\vec{w} = \langle 1, -1, 0 \rangle$ ,  $\text{proj}_{\vec{w}}(\vec{v}) = \langle 1/2, -1/2, 0 \rangle$ ,  $\text{comp}_{\vec{w}}(\vec{v}) = 1/\sqrt{2}$ .