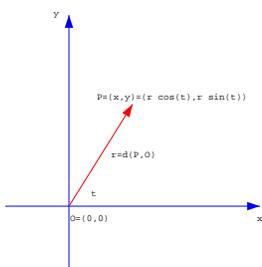


**POLAR INTEGRATION**

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**POLAR COORDINATES.** A point  $(x, y)$  in the plane has the **polar coordinates**  $r = \sqrt{x^2 + y^2}, \theta = \arctg(y/x)$ . We have  $x = r \cos(\theta), y = r \sin(\theta)$ .

Footnote: Note that  $\theta = \arctg(y/x)$  defines the angle  $\theta$  only up to an addition of  $\pi$ . The points  $(x, y)$  and  $(-x, -y)$  would have the same  $\theta$ . In order to get the correct  $\theta$ , one could take  $\arctan(y/x)$  in  $(-\pi/2, \pi/2]$  as Mathematica does, where  $\pi/2$  is the value when  $y/x = \infty$ , and add  $\pi$  if  $x < 0$  or  $x = 0, y < 0$ . In Mathematica, you can get the polar coordinates with  $(r, \theta) = (\text{Abs}[x + Iy], \text{Arg}[x + Iy])$ .



**POLAR CURVES.** A general polar curve is written as  $(r(t), \theta(t))$ . It can be translated into  $x, y$  coordinates:  $x(t) = r(t) \cos(\theta(t)), y(t) = r(t) \sin(\theta(t))$ .

**POLAR GRAPHS.** Curves which are graphs when written in polar coordinates are called **polar graphs**.

EXAMPLE.  $r(\theta) = \cos(3\theta)$  is the which belongs to the class of **roses**  $r(t) = \cos(nt)$ .

EXAMPLE. If  $y = 2x + 3$  is a line, then the equation gives  $r \sin(\theta) = 2r \cos(\theta) + 3$ . Solving for  $r(t)$  gives  $r(\theta) = 3/(\sin(\theta) - 2 \cos(\theta))$ . The line is also a polar graph.

EXAMPLE. The polar form  $r(\theta) = \frac{a(1-\epsilon^2)}{1+\epsilon \cos(\theta)}$  of the ellipse (see Kepler). The ellipse is a polar graph.

**POLAR COORDINATES.** For many regions, it is better to use polar coordinates for integration:

$$\int \int_R f(x, y) \, dx dy = \int \int_R g(r, \theta) r \, dr d\theta$$

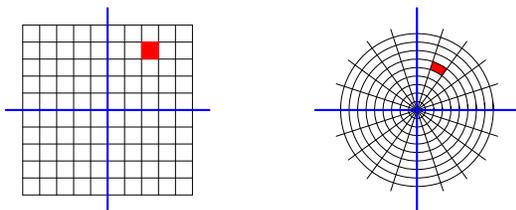
For example if  $f(x, y) = x^2 + x^2 + xy$ , then  $g(r, \theta) = r^2 + r^2 \cos(\theta) \sin(\theta)$ .

EXAMPLE. We had computed area of the disc  $\{x^2 + y^2 \leq 1\}$  using substitution. Was quite a mess. It is easier to do that integral in polar coordinates:

$$\int_0^{2\pi} \int_0^1 r \, dr d\theta = 2\pi r^2/2|_0^1 = \pi.$$

**WHERE DOES THE FACTOR "r" COME FROM?**

A small rectangle  $R$  with dimensions  $d\theta dr$  in the  $(r, \theta)$  plane is mapped by  $T: (r, \theta) \mapsto (r \cos(\theta), r \sin(\theta))$  to a sector  $S$  in the  $(x, y)$  plane. It has approximately the area  $r d\theta dr$ . It is small for small  $r$ .

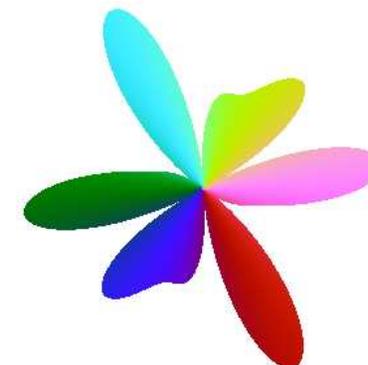
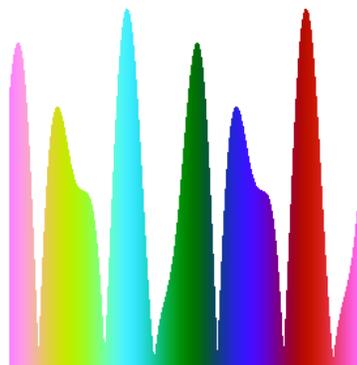


**AN OTHER EXPLANATION** (for people who have seen some linear algebra). The map translating from polar coordinates to Cartesian coordinates  $(x, y) = T(r, \theta) = (r \cos(\theta), r \sin(\theta))$  has as a linear approximation the matrix

$$DT(x, y) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r \sin(\theta) & r \cos(\theta) \end{bmatrix}$$

which has determinant  $r$ . A small rectangle in the  $(r, \theta)$ -plane of area  $dA$  will have in the  $(x, y)$  plane the area  $r dA$ .

**ROSES.** We can now integrate over type I or type II regions in the  $(\theta, r)$  plane. Examples are **roses**:  $\{(\theta, r) \mid 0 \leq r \leq f(\theta)\}$  where  $f(\theta)$  is a periodic function of  $\theta$ .



The region  $R$  in the  $\theta - r$  coordinates is a type I region

The region  $S = T(R)$  in the  $x - y$  coordinates is neither a type I nor a type II region.

EXAMPLE. Find the area of the region  $\{(\theta, r(\theta)) \mid r(\theta) \leq |\cos(3\theta)|\}$ .

$$\int \int_R y \, dx dy = \int_0^{2\pi} \int_0^{|\cos(3\theta)|} r \, dr d\theta = \int_0^{2\pi} \frac{\cos(3\theta)^2}{2} d\theta = \pi/2$$

EXAMPLE. Integrate  $f(x, y) = y\sqrt{x^2 + y^2}$  over the semi annulus  $R = \{(x, y) \mid 1 < x^2 + y^2 < 4, y > 0\}$ .

Solution.

$$\int_1^2 \int_0^\pi r \sin(\theta) r \, r \, d\theta dr = \int_1^2 r^3 \int_0^\pi \sin(\theta) \, d\theta dr = \frac{(2^4 - 1^4)}{4} \int_0^\pi \sin(\theta) \, d\theta = 15/2$$

For integration problems, where the region is part of an annulus, or if you see function with terms  $x^2 + y^2$  try to use polar coordinates  $x = r \cos(\theta), y = r \sin(\theta)$ .

**THE SUPER-CURVE.** The Belgian Biologist Johan Gielis came up in 1997 with the family of curves given in polar coordinates as

$$r(\phi) = \left( \frac{|\cos(\frac{m\phi}{4})|^{n_1}}{a} + \frac{|\sin(\frac{m\phi}{4})|^{n_2}}{b} \right)^{-1/n_3}$$

```
S[m_, n1_, n2_, n3_, a_, b_] := Module[{t},
  t := t + Abs[Cos[m t/4]/a]^n1;
  t := t + Abs[Sin[m t/4]/b]^n2;
  R[t_] := (t1 + t2/t)^(-1/n3);
  R[t_] := If[R[t] == 0, 0, R[t]];
  ParametricPlot[t, {t, -2Pi, 2Pi}, PlotRange -> All,
  PlotPoints -> 1000, Axes -> False, AspectRatio -> 1, Frame -> False];
];
```

It is called the **super-curve** because it can produce a variety of shapes like circles, square, triangle, stars. It can also be used to produce "super-shapes" (see later).

The super-curve generalizes the **superellipse** which had been discussed in 1818 by Lamé and helps to tackle one of the more intractable problems in biology: describing form. A twist: Gielis has patented his discovery described in "Gielis, J. A 'generic geometric transformation that unifies a wide range of natural and abstract shapes'. American Journal of Botany, 90, 333 - 338, (2003). To the right you see the Mathematica code.

