Extended hour to hour syllabus

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1. Week: Geometry and Space

27. June: Space, coordinates, distance

Class starts with a short slide show highlighting some points of the syllabus and the material which waits for you. Then we dive right into the subject. The idea to use **coordinates** to describe space was promoted by **René Descartes** in the 16'th century at about the time, when **Harvard College** was founded. A fundamental notion is the **distance** between two points. Pythagoras theorem allows to measure a concrete distance in some Bostonian unit. In order to get a feel about space, we will look at some geometric objects defined by coordinates. We will focus on **circles** and **spheres** and learn how to find the midpoint and radius of a sphere given as a quadratic expression in x, y, z. This method is called **completion of the square**. We will discuss, what distinguishes Euclidian distance from other distances. An other more philosophical question is why our physical space is three-dimensional. A further topic for discussion is the existence of other coordinate systems like the photographers coordinate system. Finally, we might mention GPS as an application of distance measurement or the open problem to find a **perfect cube**, a cube which the length of all sides, side diagonals as well as space diagonals are integers and which will be a homework problem ...

28. June: Vectors, dot product, projections

Two points P, Q define a **vector** $\vec{PQ}$. This includes the case $P = Q$, where $\vec{PQ}$ is the **null vector**. The vector connects the initial point P with the end point Q. Vectors can be attached everywhere in space but are identified if they have the same length and direction. Vectors can describe for example **velocities**, **forces** or **color** or **data**. We learn first algebraic operations of vectors like **addition**, **subtraction** and **scaling**. This is done both graphically as well as algebraically. We introduce then the **dot product** between two vectors which results in a scalar. Using the dot product, we can compute **length**, **angles** and **projections**. By assuming the trigonometric cos-formula, we prove the important formula $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos \alpha$, which relates length and angle with the dot product. This formula has some consequences like the **Cauchy-Schwartz inequality** or the **Pythagoras theorem**. We mention the notation $\vec{i}, \vec{j}, \vec{k}$ for the unit vectors.

29. June: Cross product, lines

The third and last lecture of the first week deals with the **cross product** of two vectors in space. The result of this product in a new vector perpendicular to both. The product can be used for many things. It is useful for example to compute **areas**, it can be used to compute the **distance** between a point and a line. It will also be important for **constructions** like to get a plane through three points or to find the line which is in the intersection of two planes. The cross product is introduced as a determinant. We will prove the important formula $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$ and interpret it geometrically as an area of the parallelepiped spanned by $\vec{v}$ and $\vec{w}$. In general, there are different ways to describe a geometric object. For lines, we will see a **parametric description**, as well as an **implicit description**. The later **symmetric equation** will later be identified as the intersection of two planes. The simplest equations are **linear equations**. A linear equation $ax + by + cz = c$ in three variables geometrically defines a **plane**. This equation can be written as $a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$ where (x_0, y_0, z_0) is a point on the plane. The equation can be interpreted as the place which is perpendicular to the vector $\vec{n} = \langle a, b, c\rangle$. We will then learn how to visualize a plane using **traces** and **intercepts**. A basic construction is to find the equation of a plane which passes through three points P,Q, and R. As an application, we look at some **distance formulas** like the **distance from a point to a plane**, the **distance from a point to a line** as well as the **the distance between two lines**. These distance formulas are geometrically useful. They illustrate how one can use the dot and cross product to measure in space.

2. Week: Functions and Surfaces

4. July: holiday

5. July: Functions, graphs, quadrics

As the name "multi-variable calculus" suggests, functions of several variables play an essential role in this course. In multivariable calculus, the focus is on functions of two or three variables. The **graph** of a function $f(x, y)$ of two variables is defined as the set of points (x, y, z) for which $z - f(x, y) = 0$. It is an example of a surface. After reviewing some **conic sections** in the plane like **hyperbola** and **parabola**, we will also look at implicit surfaces of the form $g(x, y, z) = 0$, where g is a function which only involves quadratic terms. Surfaces of this type are called **quadrics**. Important quadrics are **spheres**, **ellipsoids**, **cones**, **cylinders**, **paraboloids** and **hyperboloids**. You will have to know the names of these animals in the zoo of functions.

6. July: Implicit and parametric surfaces

Surfaces can be described in two fundamental ways: implicitly or parametrically. The **implicit description** is $g(x, y, z) = 0$ like the sphere $x^2 + y^2 + z^2 - 1 = 0$, the **parametric description** is $r(u, v) = (x(u, v), y(u, v), z(u, v))$ like $r(u, v) = (r\cos(u)\sin(v), r\sin(u)\sin(v), r\cos(v))$ In many cases, it is possible to go from one form to the other. There are four important types of surfaces for which one can do that: spheres or ellipsoids, planes, graphs of functions of two variables and surfaces of revolution. Using a computer, one can **visualize** surfaces very well. Computer algebra systems with graphical capabilities can help to do so. These tools are for the mathematician what the telescope is for the astronomer or what the microscope is for the biologist. With a bit of patience, you find your own surface which nobody has seen before and which will bear your name at the end of the course.

3. Week: Curves and Partial Derivatives

11. July: Curves, velocity, acceleration, chain rule

Curves are one-dimensional objects. One can look at curves both in the plane as well as in space, they can take many different shapes. A special case are **closed curves** in space which are called **knots**. We will learn how to describe curves by parametrization $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$. By differentiation, one obtains the **velocity** $\vec{r}\,'(t)$ and the **acceleration** $\vec{r}\,''(t)$, which are both vectors. The **speed** of a curve at some point is the length $|\vec{r}\,'(t)|$ of the velocity vector. The usual **one dimensional chain rule** $\frac{d}{ds}\vec{r}(t(s)) = \vec{r}(t(s))t'(s)$ gives the volocity after a change of time.

12. July: Arc-length, curvature, partial derivatives

We first derive the **arc length formula** for of a curve. This length can be computed by evaluating a one-dimensional integral, by integrating up the **speed** from the start time to the stop time. The **curvature** of a curve is a quantity telling how much a curve is bent. A circle of radius r has the curvature $1/r$. Finally, we will see partial derivatives as well as see some **partial differential equations** (PDE)'s. An example of a partial differential equation is the wave equation $u_{tt} = u_{xx}$ which describes sound or light waves.

13. July: First midterm (on week 1-2)

4. Week: Extrema and Lagrange Multipliers

18. July: Gradient, linearization, tangents, chain rule

The **gradient** of a function is an important tool to describe the geometry of surfaces. Fundamental is the property that the gradient vector ∇g is perpendicular to the implicit surface $g = c$. This allows us to compute **tangent planes** and **tangent lines** as well as to approximate a linear function by a linear function near a point. Many physical laws are actually just linearization of more complicated nonlinear laws.

19. July: Extrema, second derivative test

A central application of multi-variable calculus is to **extremize** functions of two variables. One first identifies **critical points**, points where the gradient vanishes. The nature of these critical points can be established using the **second derivative test**. There will be three fundamentally different cases: **local maxima**, **local minima** as well as **saddle points**.

20. July: Extrema with constraints

The topic with maybe the most applications both in science or economics is to extremize a function $f(x, y)$ in the presence of a **constraint** $g(x, y) = 0$. A necessary condition for a critical point is that the gradients of f and g are parallel. This leads to equations called the **Lagrange equation**.

5. Week: Double Integrals and Surface Integrals

25. July: Double integrals, type I,II regions

Integration in two dimensions is first done on rectangles, then on regions bound by graphs of functions. Similar than in one dimension, there is a **Riemann sum approximation** of the integral. This allows us to prove results like **Fubinis theorem** on the change of the integration order. An application of double integration is the computation of **area**.

26. July: Polar coordinates, surface area

Many regions can be described better in **polar coordinates**. Examples are so called **roses** which trace flower-like shapes in the plane but are graphs in polar coordinates. Changing coordinates comes with an integration factor which can be explained also after introducing the surface area.

27. July: Second midterm (on week 3-4)

Triple Integrals and Line Integrals

1. August: Triple integrals, cylindrical coordinates

Triple integrals allow the computation of volumes, moment of inertias or centers of masses of solids. First introduced for cubes it is then extended to more general regions bound by graphs of functions of two variables. Some regions can be described better in **cylindrical coordinates**, the analogue of polar coordinates in space.

2. August: Spherical coordinates, vector fields

Spherical coordinates allow an even more elegant computation of triple integrals for certain regions like cones or spheres. Next, we will introduce **vector fields**. They occur as force fields or velocity fields or mechanics and are closely related to the field of ordinary differential equations. Vector fields will occupy us until the end of the course.

3. August: Line integrals, fundamental thm of line integrals

Line integrals are one dimensional integrals along a curve in the presence of a vector field. If the vector field is a force field, then the line integral has the interpretation work done, when walking along the path. For a class of vector fields which we call **conservative vector fields** one can compute the line integral easily using an identity called the fundamental theorem of line integrals.

Exterior Derivatives and Integral Theorems

8. August: Curl and Green theorem

Greens theorem relates a line integral along a closed curve with a double integral of a derivative of the vector field in the region enclosed by the curve. The theorem is useful for example to compute areas. It also allows an easy computation of line integrals in certain cases. We will see a derivative of the vector field which is called the **"curl"**. It is a scalar field which measures the **vorticity** of the vector field in the plane.

9. August: Curl and Stokes theorem

Stokes theorem is Greens theorem lifted into three dimensions, where the region is replaced by a surface. Again, one can replace the line integral along the boundary of the surface by an integral of the "curl" of the field over the surface. This integral is a **flux integral**. The curl of a vector field in three dimensions is a vector field itself. The three components give the vorticity of the vector field in the x,y and z direction.

10. August: Div and Gauss theorem

Finally, the **divergence** of a vector field inside a solid is related to the flux of the vector field through the boundary of the surface using the **divergence theorem** which is sometimes also called **Gauss theorem**. The divergence theorem relates the "local expansion rate" of a vector field with the flux through a closed surface and is useful for example to compute the gravitational field inside a solid.

15. August: Final exam (on week 1-7) 1:30 PM