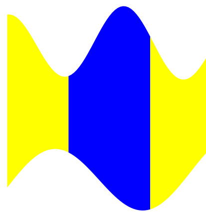
2D INTEGRALS FOR GENERAL REGIONS — Maths21a, O. Knill

TYPE I REGIONS. A **type I region** is bound between the graphs of two functions $c(x)$ and $d(x)$. One can write the region as

$$R = \{(x,y) \mid c(x) \leq y \leq d(x)\ \} .$$

An integral over such a region is an iterated integral:

$$\iint_R f\ dA = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\ dydx$$

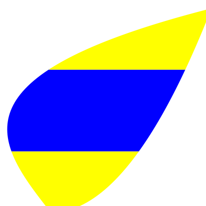


TYPE II REGIONS. A type I region turned by 90 degrees is called a type II region. It is defined by two functions $a(y)$ and $b(y)$ which are functions of y. One can write the region as

$$R = \{(x,y) \mid a(y) \leq x \leq b(y)\ \} .$$

An integral over such a region is an iterated integral:

$$\iint_R f\ dA = \int_c^d \int_{a(y)}^{b(y)} f(x,y)\ dxdy$$



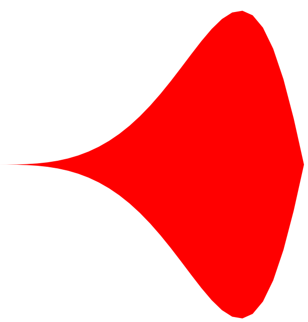
RECTANGLES. Rectangles are both type I and type II regions. For rectangles, the functions $a(y), b(y), c(x)$ and $d(x)$ are all constant.

EXAMPLE 1) Integrate $f(x,y) = x^2$ over the region bounded above by $\sin(x^3)$ and bounded below by the grapy of $-\sin(x^3)$ for $0 \leq x \leq \pi$. The value of this integral has a physical meaning. It is a moment of inertia. We will come back to that next week.

$$\int_0^{\pi^{1/3}} \int_{-\sin(x^3)}^{\sin(x^3)} x^2\ dydx = 2\int_0^{\pi^{1/3}} \sin(x^3)x^2\ dx =$$

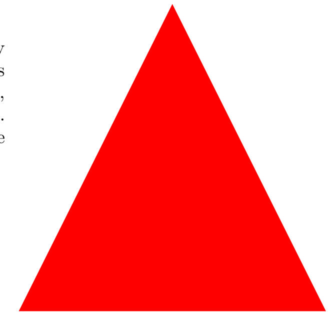
Now, we have an integral, which we can solve by substitution

$$= -(2/3)\cos(x^3)|_0^{\pi^{1/3}} = 4/3$$



EXAMPLE 2) Integrate $f(x,y) = y^2$ over the region bound by the $x-$ axes, the lines $y = x+1$ and $y = 1-x$. The problem is best solved as a type II integral. As you can see from the picture, we would have to compute 2 different integrals as a type I integral. To do so, we have to write the bounds as a function of y: they are $x = y-1$ and $x = 1-y$

$$\int_0^1 \int_{y-1}^{1-y} y^3\ dx\ dy = 2\int_0^1 y^3(1-y)\ dy = 2(1/4-1/3) = 1/10 .$$



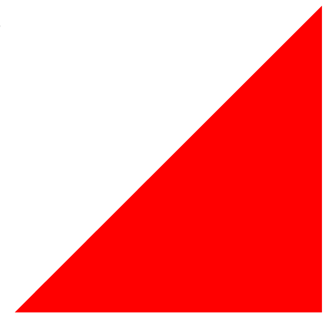
EXAMPLE. Let R be the triangle $1 \geq x \geq 0, 0 \leq y \leq x$. What is

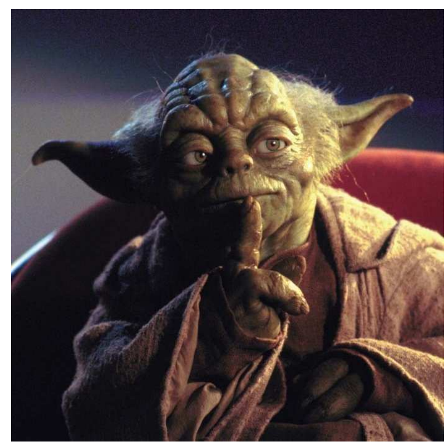
$$\int\int_R e^{-x^2}\ dxdy\ ?$$

The type II integral $\int_0^1 [\int_y^1 e^{-x^2}\ dx]dy$ can not be solved because e^{-x^2} has no anti-derivative in terms of elementary functions.

The type I integral $\int_0^1 [\int_0^x e^{-x^2}\ dy]\ dx$ however can be solved:

$$= \int_0^1 xe^{-x^2}\ dx = -\frac{e^{-x^2}}{2}|_0^1 = \frac{(1-e^{-1})}{2} = 0.316... .$$





WORDS OF WISDOM:

If a double integral you can not solve, the order of integration change you must.

For solving double integrals, a picture at hand must be.