FUNDAMENTAL THM OF LINE INTEGRALS — Maths21a, O. Knill

LINE INTEGRALS. If F is a vector field and $C : t \mapsto \vec{r}(t)$ is a curve, then $\boxed{\int_a^b F(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt}$ is called the **line integral** of F along the curve C.

GRADIENT FIELD. A vector field F is called a **gradient field** if there exists a function f such that $F = \nabla f$. For example, $F(x,y) = (xy^2, yx^2)$ is a gradient field.

FUNDAMENTAL THEOREM OF LINE INTEGRALS. If $F = \nabla f$, then

$$\boxed{\int_a^b F(r(t)) \cdot r'(t)\; dt = f(r(b)) - f(r(a))}$$

In other words, the line integral integral is the potential difference between the end points $\vec{r}(b)$ and $\vec{r}(a)$, if F is a gradient field.

EXAMPLE. Let $f(x,y,z)$ be the temperature distribution in a room and let $\vec{r}(t)$ the path of a fly in the room, then $f(\vec{r}(t))$ is the temperature, the fly experiences at the point $\vec{r}(t)$ at time t. The change of temperature for the fly is $\frac{d}{dt} f(\vec{r}(t))$. The line-integral of the temperature gradient ∇f along the path of the fly coincides with the temperature difference between the end and initial point.

SPECIAL CASES.
If $\vec{r}(t)$ is parallel to the level curve of f, then $d/dt f(\vec{r}(t)) = 0$ and $\vec{r}\,'(t)$ orthogonal to $\nabla f(\vec{r}(t))$ If $\vec{r}(t)$ is orthogonal to the level curve, then $|d/dt f(\vec{r}(t))| = |\nabla f||\vec{r}\,'(t)|$ and $\vec{r}\,'(t)$ is parallel to $\nabla f(\vec{r}(t))$.

PROOF OF THE FUNDAMENTAL THEOREM. Use the chain rule in the second equality and the fundamental theorem of calculus in the third equality of the following identities:

$$\int_a^b F(r(t)) \cdot r'(t)\; dt = \int_a^b \nabla f(r(t)) \cdot r'(t)\; dt = \int_a^b \frac{d}{dt} f(r(t))\; dt = f(r(b)) - f(r(a)) \; .$$

CONSERVATIVE. A field F is called **conservative** if every line integral is independent of paths.

> F is a gradient field if and only if it is conservative.

Proof. If $F = \nabla f$, the conservative property follows from the fundamental theorem of line integrals. To see the other direction, chose a point y and for every x a path C_x connecting y with x and define $f(x) = \int_{C_x} F \cdot dr$. The conservative property assures that the result is independent of the chosen path. Note that f is not unique: changing y will add a constant to f.

CLOSED LOOP PROPERTY = ENERGY CONSERVATION. It follows that for a gradient field the line-integral along any closed curve is zero. Conversely, if this "energy conservation" holds, one has a conservative field.

> If F is a gradient field if and only if the line integral along a closed curve is zero if and only if the field is a gradient field.

PERPETUUM MOTION MACHINES. A device which implements a force field which is not a gradient field is called a **perpetuum motion machine**. Mathematically, it realizes a force field for which along some closed loops the energy gain is nonnegative. (y possibly changing the direction, the energy change is positive. The first law of thermodynamics forbids the existence of such a machine.
It is informative to contemplate some of the ideas people have come up with and to see why they don't work. The drawings of Escher appear also to produce situations, where a force field can be used to gain energy. Escher uses genius graphical tricks however.

THE COMPONENT TEST.
When is a vector field a gradient field? $F(x,y) = \nabla f(x,y)$ implies $F_y(x,y) = F_x(x,y)$. If this does not hold at some point, $F = (P,Q)$ is no gradient field. This is called the **component test**. The condition $\text{curl}(F) = Q_x - P_y = 0$ implies that the field is conservative if the region satisfies a certain property.

FIND THE POTENTIAL.

PROBLEM 1. Let $F(x,y) = (2xy^2 + 3x^2, 2yx^2)$. Find a potential f of F.

SOLUTION. The potential function $f(x,y)$ satisfies $f_x(x,y) = 2xy^2 + 3x^2$ and $f_y(x,y) = 2yx^2$. Integrating the second equation gives $f(x,y) = x^2y^2 + h(x)$. Partial differentiation with respect to x gives $f_x(x,y) = 2xy^2 + h'(x)$ which should be $2xy^2 + 3x^2$ so that we can take $h(x) = x^3$. The potential function is $f(x,y) = x^2y^2 + x^3$.

Find g, h from $f(x,y) = \int_0^x M(x,y)\; dx + h(y)$ and $f_y(x,y) = g(x,y)$.

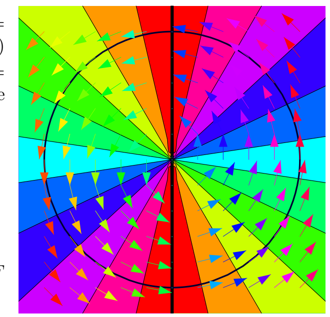
PROBLEM 2. Find values for the constants a,b which make the vector field $F = (P,Q) = (ax^3y + by^2, x^4 + yx)$ conservative.
SOLUTION. The curl $Q_x - P_y = 4x^3 + y - ax^3$ must be zero. This gives $a = 4$ and $b = 1/2$. The potential is $f(x,y) = (x^4 + y^2x/2)$.

A COUNTER EXAMPLE? Let $F(x,y) = (P,Q) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$. It is a gradient field because $f(x,y) = \arctan(y/x)$ has the property that $f_x = (-y/x^2)/(1+y^2/x^2) = M, f_y = (1/x)/(1+y^2/x^2) = N$. However, the line integral $\int_\gamma F\; ds$, where γ is the unit circle is

$$\int_0^{2\pi} (\frac{-\sin(t)}{\cos^2(t) + \sin^2(t)}, \frac{\cos(t)}{\cos^2(t) + \sin^2}) \cdot (-\sin(t), \cos(t)\; dt$$

which is $\int_0^{2\pi} 1\; dt = 2\pi$. What is wrong?

Solution: note that the potential f as well as the vector-field F are not smooth everywhere.



SIMPLY CONNECTED. A region R is called **simply connected**, if every curve in R can be contracted to a point in a continuous way and every two points can be connected by a path. A disc is an example of a simply connected region, an annulus is an example which is not.

"CONSERVATIVE $\Leftrightarrow$ $\text{curl}(F) = 0$.

> If R is a simply connected region, then F is a gradient field if and only if $\text{curl}(F) = 0$ everywhere in R.

We will prove this later.

SUMMARY.

We have three equivalent properties: conservative=gradient field= closed loop property. They imply the component test $\text{curl}(F) = 0$. If the region is simply connected, all four properties are equivalent.