GRADIENT AND DIRECTIONAL DERIVATIVE   O.knill,Maths21a

GRADIENT. For a function $f(x,y)$ of two variables, define the **gradient** $\nabla f(x,y) = (f_x(x,y), f_y(x,y))$. For a function of three variables, define $\nabla f(x,y,z) = (f_x(x,y,z), f_y(x,y,z), f_z(x,y,z))$ in three dimensions.

CHAIN RULE. The chain rule in multivariable calculus is $\boxed{\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)}$ It looks like the 1D chain rule, but the derivative f' is replaced with the gradient and the derivative of r is the velocity. Written out in detail the chain rule is

$$\frac{d}{dt} f(x(t), y(t), z(t)) = f_x(x(t), y(t), z(t))x'(t) + f_y(x(t), y(t), z(t))y'(t) + f_z(x(t), y(t), z(t))z'(t) .$$

DIRECTIONAL DERIVATIVE. If f is a function of several variables and $\vec{v}$ is a vector, then $\nabla f \cdot \vec{v}$ is called the **directional derivative** of f in the direction $\vec{v}$. One writes $\nabla_{\vec{v}} f$ or $D_{\vec{v}} f$.

$$D_v f(x,y,z) = \nabla f(x,y,z) \cdot v$$

Unlike done in some calculus books, we do not insist that $\vec{v}$ is a unit vector. The chain rule gives $\frac{d}{dt} D_{\vec{v}} f = \frac{d}{dt} f(x + t\vec{v})$.

EXAMPLE. PARTIAL DERIVATIVES ARE SPECIAL DIRECTIONAL DERIVATIVES.

If $\vec{v} = (1,0,0)$, then $D_{\vec{v}} f = \nabla f \cdot v = f_x$.   The directional derivative is a generalization of the
If $\vec{v} = (0,1,0)$, then $D_{\vec{v}} f = \nabla f \cdot v = f_y$.   partial derivatives. Like the partial derivatives, it is
If $\vec{v} = (0,0,1)$, then $D_{\vec{v}} f = \nabla f \cdot v = f_z$.   a **scalar**.

EXAMPLE. DIRECTIONAL DERIVATIVE ALONG A CURVE.
If f is the temperature in a room and $\vec{r}(t)$ is a curve with velocity $\vec{r}'(t)$, then $\nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$ is the temperature change, one measures on the point moving on a curve $\vec{r}(t)$ experiences: the chain rule told us that this is $d/dt f(\vec{r}(t))$.

GRADIENTS AND LEVEL CURVES/SURFACES.

Gradients are orthogonal to level curves and level surfaces.

Every curve $\vec{r}(t)$ on the level curve or level surface satisfies $\frac{d}{dt} f(\vec{r}(t)) = 0$. By the chain rule, $\nabla f(\vec{r}(t))$ is perpendicular to the tangent vector $\vec{r}'(t)$.
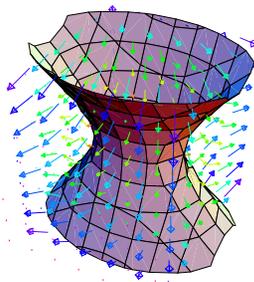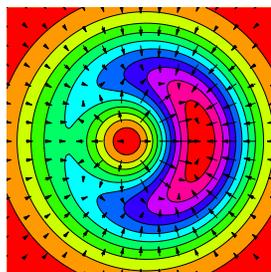
STEEPEST DECENT. The directional derivative satisfies $\boxed{|D_{\vec{v}} f| \leq |\nabla f||\vec{v}|}$ because $\nabla f \cdot \vec{v} = |\nabla f||\vec{v}||\cos(\phi)| \leq |\nabla f||\vec{v}|$. The direction $\vec{v} = \nabla f$ is the direction, where f **increases** most, the direction $-\nabla f$ is the direction where f **decreases most**. It is the direction of steepest decent.

IN WHICH DIRECTION DOES f INCREASE? If $\vec{v} = \nabla f$, then the directional derivative is $\nabla f \cdot \nabla f = |\nabla f|^2$. This means that f **increases**, if we move into the direction of the gradient!

EXAMPLE. You are on a trip in a air-ship at $(1,2)$ and want to avoid a thunderstorm, a region of low pressure. The pressure is given by a function $p(x,y) = x^2 + 2y^2$. In which direction do you have to fly so that the pressure change is largest?

Parameterize the direction by $\vec{v} = (\cos(\phi), \sin(\phi))$. The pressure gradient is $\nabla p(x,y) = (2x, 4y)$. The directional derivative in the ϕ-direction is $\nabla p(x,y) \cdot v = 2\cos(\phi) + 4\sin(\phi)$. This is maximal for $-2\sin(\phi) + 4\cos(\phi) = 0$ which means $\tan(\phi) = 1/2$.

ZERO DIRECTIONAL DERIVATIVE. The rate of change in all directions is zero if and only if $\nabla f(x,y) = 0$: if $\nabla f \neq \vec{0}$, we can choose $\vec{v} = \nabla f$ and get $D_{\nabla f} f = |\nabla f|^2$.

We will see later that points with $\nabla f = \vec{0}$ are candidates for **local maxima** or **minima** of f. Points (x,y), where $\nabla f(x,y) = (0,0)$ are called **stationary points** or **critical points**. Knowing the critical points is important to understand the function f.

PROPERTIES DIRECTIONAL DERIVATIVE.   PROPERTIES GRADIENT

$$\begin{aligned} D_v(\lambda f) &= \lambda D_v(f) \\ D_v(f + g) &= D_v(f) + D_v(g) \\ D_v(fg) &= D_v(f)g + fD_v(g) \end{aligned} \qquad \begin{aligned} \nabla(\lambda f) &= \lambda \nabla(f) \\ \nabla(f + g) &= \nabla(f) + \nabla(g) \\ \nabla(fg) &= \nabla(f)g + f\nabla(g) \end{aligned}$$

THE MATTERHORN is a popular climbing mountain in the Swiss alps. Its height is 4478 meters (14,869 feet). It is quite easy to climb with a guide. There are ropes and ladders at difficult places. Even so, about 3 people die each year from climbing accidents at the Matterhorn, this does not stop you from trying an ascent. In suitable units on the ground, the height $f(x,y)$ of the Matterhorn is approximated by $f(x,y) = 4000 - x^2 - y^2$. At height $f(-10,10) = 3800$, at the point $(-10, 10, 3800)$, you rest. The climbing route continues into the north-east direction $v = (1, -1)$. Calculate the rate of change in that direction. We have $\nabla f(x,y) = (-2x, -2y)$, so that $(20, -20) \cdot (1, -1) = 40$. This is a place, with a ladder, where you climb 40 meters up when advancing 1m forward.


THE VAN DER WAALS (1837-1923) equation for real gases is

$$(p + a/V^2)(V - b) = RT(p,V) ,$$

where $R = 8.314 J/K$mol is a constant called the **Avogadro number**. This law relates the pressure p, the volume V and the temperature T of a gas of one mol. The constant a is related to the molecular interactions, the constant b to the finite rest volume of the molecules.
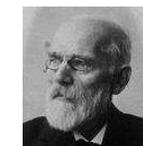
The **ideal gas** law $pV = RT$ for a gas of one mol $n = 1$ is obtained when a, b are set to 0. The level curves or **isotherms** $T(p,V) = const$ tell much about the properties of the gas. The so called **reduced van der Waals law** $T(p,V) = (p + 3/V^2)(3V - 1)/8$ is obtained by scaling p, T, V depending on the gas. Calculate the directional derivative of $T(p,V)$ at the point $(p,V) = (1,1)$ into the direction $v = (1,2)$. We have $T_p(p,V) = (3V - 1)/8$ and $T_V(p,V) = 3p/8 - (9/8)1/V^2 - 3/(4V^3)$. Therefore, $\nabla T(1,1) = (1/4, 0)$ and $D_v T(1,1) = 1/5$.
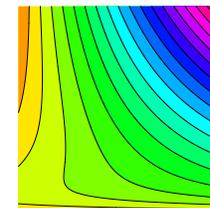
TANGENT LINE. Because $\vec{n} = \nabla f(x_0, y_0) = \langle a, b \rangle$ is perpendicular to the level curve $f(x,y) = c$ through (x_0, y_0), the equation for the tangent line is

$$\boxed{ax + by = d, \ a = f_x(x_0, y_0), \ b = f_y(x_0, y_0), \ d = ax_0 + by_0}$$

EXAMPLE. The isotherm in the previous example through $(1,1)$ has there the tangent $(1/4)x + 0 \cdot y = (1/4)1 + 0 \cdot 1 = 1/4$ which is the horizontal line $x = 1$.