

CYLINDRICAL COORDINATES

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REMINDER: INTEGRATION POLAR COORDINATES.

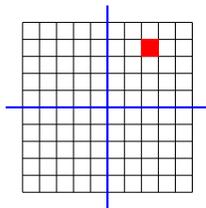
$$\int \int_R f(r, \theta) \boxed{r} d\theta dr .$$

EXAMPLE 1. Area of a disk of radius R

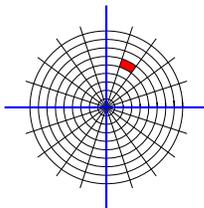
$$\int_0^R \int_0^{2\pi} r \, d\theta dr = 2\pi \frac{r^2}{2} \Big|_0^R = R^2 \pi .$$

WHERE DOES THE FACTOR "r" COME FROM?

EXPLANATION. A small rectangle with dimensions $d\theta dr$ in the (r, θ) plane is mapped to a sector segment in the (x, y) plane. It has approximately the area $r d\theta dr$. It is small for small r .



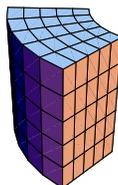
$$T : (r, \theta, z) \mapsto (r \cos(\theta), r \sin(\theta), z)$$



CYLINDRICAL COORDINATES. Use polar coordinates in the x - y plane and leave the z coordinate. Take $T(r, \theta, z) = (r \cos(\theta), r \sin(\theta), z)$. The integration factor r is the same as in polar coordinates.

$$\int \int \int_{T(R)} f(x, y, z) \, dx dy dz = \int \int \int_R g(r, \theta, z) \boxed{r} \, dr d\theta dz .$$

For example, if $f(x, y, z) = (x^2 + y^2) + xz$, then $g(r, \theta, z) = r^2 + r \cos(\theta)z$.



EXAMPLE. Calculate the volume bounded by the parabolic $z = 1 - (x^2 + y^2)$ and the x - y plane. In cylindrical coordinates, the paraboloid is given by the relation $z = 1 - r^2$:

$$\int_0^1 \int_0^{2\pi} \int_0^{1-r^2} r \, dz d\theta dr = \int_0^1 \int_0^{2\pi} (r - r^3) \, d\theta dr = 2\pi(r^2/2 - r^4/4) \Big|_0^1 = \pi .$$

USE A GOOD PICTURE! A good conceptual picture not only helps to solve double and triple integral problems. Sometimes, it is even virtually impossible to solve the problem without having a good picture.

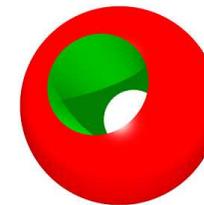
PROBLEM. Find the volume of the solid obtained by taking a sphere $x^2 + y^2 + z^2 = 1$ into which a hole $x^2 + y^2 \leq 1/2$ has been drilled.

SOLUTION.

$$2\pi \int_{1/2}^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r dz dr = 2\pi \int_{1/2}^1 2r\sqrt{1-r^2} \, dr$$

which is

$$-2\pi \frac{2}{3} (1-r^2)^{3/2} \Big|_{1/2}^1 = \frac{4\pi}{3} \frac{\sqrt{27}}{8} = \pi\sqrt{3}/2$$

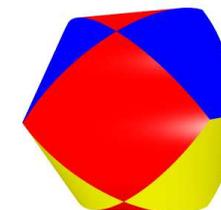
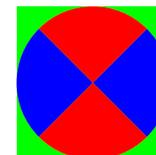


PROBLEM. Find the volume of the intersection of the three cylinders $x^2 + y^2 \leq 1$, $x^2 + z^2 \leq 1$ and $y^2 + z^2 \leq 1$.

SOLUTION.

$$8 \int_{-\pi/4}^{\pi/4} \int_0^1 \sqrt{1-r^2} \cos(\theta)^2 \, r dr d\theta$$

which is $16 - 8\sqrt{2}$.



PROBLEM. Find the volume of the intersection of the two solid cylinders $x^2 + y^2 \leq 1$ and $x^2 + z^2 \leq 1$.

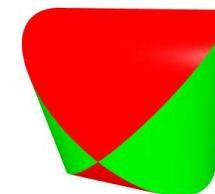
SOLUTION. We compute the volume in one of the 8 octants and multiply by 8 in the end.

$$8 \int_0^1 \int_0^{\pi/2} \sqrt{1-r^2} \cos(\theta)^2 \, r d\theta dr = 16/3 .$$

Here is how one would evaluate this integral with Mathematica:

8 Integrate[Sqrt[1 - r^2 Cos[theta]^2] r, {r, 0, 1}, {theta, 0, Pi/2}]

Note the order in which the integration range in entered the computer algebra system!



PROBLEM. Find $\int \int \int_R z^2 \, dV$, where R of the solid obtained by intersecting $\{1 \leq x^2 + y^2 + z^2 \leq 4\}$ with the double cone $\{z \geq x^2 + y^2\}$.

SOLUTION. We split the integral up into a "cone part" $z \in [-\sqrt{2}, \sqrt{2}]$, and the cup part $|z| > \sqrt{2}$ and evaluate each separately. The double cone has the volume $2\pi \int_{-\sqrt{2}}^{\sqrt{2}} \int_0^z r \, dr dz = 2\pi\sqrt{2}/3$. One cup has the volume $2\pi \int_{\sqrt{2}}^2 \int_0^{\sqrt{4-z^2}} r \, dr dz = \pi \int_{\sqrt{2}}^2 (4 - z^2) \, dz$. The total volume is $\pi(2 - \sqrt{2})16/3$.

