CURL (3D). The curl of a 3D vector field $F = (P, Q, R)$ is defined as the 3D vector field

$$\mathrm{curl}(P, Q, R) = (R_y - Q_z, P_z - R_x, Q_x - P_y).$$

CURL (2D). Recall, the curl of a 2D vector field $F = (P, Q)$ is $Q_x - P_y$, a scalar field.
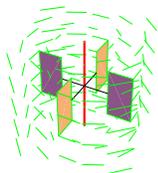
WANTED! Is there a multivariable calculus book, in which the above wheel is **not** shown? The wheel indicates the curl vector if F is thought of as a wind velocity field. As we will see later the direction in which the wheel turns fastest, is the direction of $\mathrm{curl}(F)$. The wheel could actually be used to measure the curl of the vector field at each point. In situations with large vorticity like in a tornado, one can "see" the direction of the curl.

DIV (3D). The **divergence** of $F = (P, Q, R)$ is the scalar field $\mathrm{div}(P, Q, R) = \nabla \cdot F = P_x + Q_y + R_z$.
DIV (2D). The **divergence** of a vector field $F = (P, Q)$ is $\mathrm{div}(P, Q) = \nabla \cdot F = P_x + Q_y$.

NABLA CALCULUS. With the "vector" $\nabla = (\partial_x, \partial_y, \partial_z)$, we can write $\mathrm{curl}(F) = \nabla \times F$ and $\mathrm{div}(F) = \nabla \cdot F$. This is both true in 2D and 3D.

LAPLACE OPERATOR. $\Delta f = \mathrm{divgrad}(f) = f_{xx} + f_{yy} + f_{zz}$ can be written as $\nabla^2 f$ because $\nabla \cdot (\nabla f) = \mathrm{div}(\mathrm{grad}(f))$. One can extend this to vectorfields $\Delta F = (\Delta P, \Delta Q, \Delta R)$ and writes $\nabla^2 F$.

IDENTITIES. While direct computations can verify the identities to the left, they become evident with Nabla calculus from formulas for vectors like $\vec{v} \times \vec{v} = \vec{0}, \vec{v} \cdot \vec{v} \times \vec{w} = 0$ or $u \times (v \times w) = v(u \cdot w) - (u \cdot v)w$.

|	
$\mathrm{div}(\mathrm{curl}(F)) = 0$.	$\nabla \cdot \nabla \times F = 0$.
$\mathrm{curlgrad}(F) = \vec{0}$	$\nabla \times \nabla F = \vec{0}$.
$\mathrm{curl}(\mathrm{curl}(F)) = \mathrm{grad}(\mathrm{div}(F) - \Delta(F))$.	$\nabla \times \nabla \times F = \nabla(\nabla \cdot F) - (\nabla \cdot \nabla)F$.

QUIZZ. Is there a vector field G such that $F = (x + y, z, y^2) = \mathrm{curl}(G)$? Answer: no because $\mathrm{div}(F) = 1$ is incompatible with $\mathrm{div}I\mathrm{curl}(G)) = 0$.

ADDENDA TO GREEN'S THEOREM. Green's theorem, one of the advanced topics in this course is useful in physics. We have already seen the following applications

- Simplify computation of double integrals
- Simplify computation of line integrals
- Formulas for centroid of region
- Formulas for area
- Justifying, why mechanical integrators like the planimeter works.

APROPOS PLANIMETER.

The **cone planimeter** is a mechanical instrument to find the antiderivative of a function $f(x)$. It uses the fact that the vector field $F(x, y) = (y, 0)$ has $\mathrm{curl}(F) = -1$. By Greens theorem, the line integral around the type I region bounded by 0 and the graph of $f(x)$ in the counter clockwise direction is $\int_a^x f(x)\ dx$. The planimeter determines that line integral.
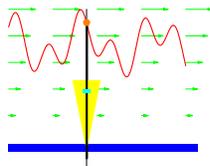
THERMODYNAMICS. Gases or liquids are often described in a $P - V$ **diagram**, where the volume in the x-axis and the pressure in the y axis. A periodic process like the pump in a refrigerator leads to closed curve $\gamma : r(t) = (V(t), p(t))$ in the $V - p$ plane. The curve is parameterized by the time t. At a given time, the gas has volume $V(t)$ and a pressure $p(t)$. Consider the vector field $F(V, p) = (p, 0)$ and a closed curve γ. What is $\int_\gamma F\ ds$? Writing it out, we get $\int_0^t (p(t), 0) \cdot (V'(t), p'(t)\ dt = \int_0^t p(t)V'(t)dt = \int_0^t pdV$. The curl of $F(V, p)$ is -1. You see by Green's theorem the integral $-\int_0^t pdV$ is the area of the region enclosed by the curve.

MAXWELL EQUATIONS (in homework you assume no current $j = 0$ and charges $\rho = 0$. c = speed of light.

$\mathrm{div}(B) = 0$	No monopoles	there are no magnetic monopoles.
$\mathrm{curl}(E) = -\frac{1}{c}B_t$	Faraday's law	change of magnetic flux induces voltage
$\mathrm{curl}(B) = \frac{1}{c}E_t + \frac{4\pi}{c}j$	Ampère's law	current or change of E produces magnetic field
$\mathrm{div}(E) = 4\pi\rho$	Gauss law	electric charges are sources for electric field

2D MAXWELL EQUATIONS? In space dimensions d different than 3 the electromagnetic field has $d(d + 1)/2$ components. In 2D, the magnetic field is a scalar field and the electric field $E = (P, Q)$ a vector field. The 2D Maxwell equations are $\mathrm{curl}(E) = -\frac{1}{c}\frac{d}{dt}B, \mathrm{div}(E) = 4\pi\rho$. Consider a region R bounded by a wire γ. Green's theorem tells us that $d/dt \int \int_R B(t)\ dxdy$ is the line integral of E around the boundary. But $\int_\gamma Eds$ is a voltage. A change of the magnetic field produces a voltage. This is the **dynamo** in 2D. We will see the real dynamo next week in 3D, where electromagnetism works (it would be difficult to generate a magnetic field in flatland). If v is the velocity distribution of a fluid in the plane, then $\omega(x, y) = \mathrm{curl}(v)(x, y)$ is the **vorticity** of the fluid. The integral $\int \int_R \omega dxdy$ is called the **vortex flux** through R. Green's theorem assures that this flux is related to the circulation on the boundary.

FLUID DYNAMICS. v velocity, ρ density of fluid.

Continuity equation	$\dot{\rho} + \mathrm{div}(\rho v) = 0$	no fluid get lost
Incompressibility	$\mathrm{div}(v) = 0$	incompressible fluids, in 2D: $v = \mathrm{grad}(u)$
Irrotational	$\mathrm{curl}(v) = 0$	irrotation fluids

A RELATED THEOREM. If we rotate the vector field $F = (P, Q)$ by 90 degrees $= \pi/2$ we get a new vector field $G = (-Q, P)$. The integral $\int_\gamma F \cdot ds$ becomes a **flux** $\int_\gamma G \cdot dn$ of G through the boundary of R, where dn is a normal vector with the length of dr. With $\mathrm{div}(F) = (P_x + Q_y)$, we see that $\mathrm{curl}(F) = \mathrm{div}(G)$. Green's theorem now becomes

$$\int\int_R \mathrm{div}(G)\ dxdy = \int_\gamma G \cdot dn,$$

where $dn(x, y)$ is a normal vector at (x, y) orthogonal to the velocity vector $r'(x, y)$ at (x, y). This new theorem has a generalization to three dimensions, where it is called Gauss theorem or divergence theorem. Don't treat this however as a different theorem in two dimensions. **It is in two dimensions just Green's theorem disguised.** There are only 2 basic integral theorems in the plane: Green's theorem and the FTLI.

PREVIEW. Green's theorem is of the form $\int_R F' = \int_{\delta R} F$, where F' is a "derivative" and δR is a "boundary". There are 2 such theorems in dimensions 2, three theorems in dimensions 3, four in dimension 4 etc. In the plane, Green's theorem is the second one besides the fundamental theorem of line integrals FTLI. In three dimensions, there are two more theorems beside the FTLI: Stokes and Gauss Theorems which we will see in the next week.

dim	dim(R)	theorem
1D	1	Fund. thm of calculus

dim	dim(R)	theorem
2D	1	Fund. thm of line integrals
2D	2	Green's theorem

dim	dim(R)	theorem
3D	1	Fundam. thm of line integrals
3D	2	Stokes theorem
3D	3	Gauss theorem

$1 \mapsto 1$	f'	derivative
$1 \mapsto 2$	∇f	gradient
$2 \mapsto 1$	$\nabla \times F$	curl

$1 \mapsto 3$	∇f	gradient
$3 \mapsto 3$	$\nabla \times F$	curl
$3 \mapsto 1$	$\nabla \cdot F$	divergence