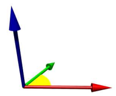
CROSS PRODUCT — Maths21a, O. Knill

CROSS PRODUCT. The **cross product** of two vectors $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ is defined as the vector $\vec{v} \times \vec{w} = \langle v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \rangle$. To compute it, look at the "determinant computation":



$$\begin{bmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \begin{bmatrix} i & & \\ & v_2 & v_3 \\ & w_2 & w_3 \end{bmatrix} - \begin{bmatrix} & j & \\ v_1 & & v_3 \\ w_1 & & w_3 \end{bmatrix} + \begin{bmatrix} & & k \\ v_1 & v_2 & \\ w_1 & w_2 & \end{bmatrix} = i(v_2 w_3 - v_3 w_2) - j(v_1 w_3 - v_3 w_1) + k(v_1 w_2 - v_2 w_1).$$

DIRECTION OF $\vec{v} \times \vec{w}$: $\boxed{\vec{v} \times \vec{w} \text{ is orthogonal to } \vec{v} \text{ and orthogonal to } \vec{w}.}$

Proof. Check that $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$.

LENGTH: $\boxed{|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)}$

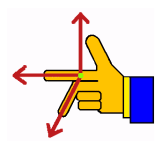
Proof. The identity $|\vec{v} \times \vec{w}|^2 = |\vec{v}|^2 |\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2$ can be proven by direct computation. Now, $|\vec{v} \cdot \vec{w}| = |\vec{v}||\vec{w}| \cos(\alpha)$.

AREA. The length $|\vec{v} \times \vec{w}|$ is the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$.

Proof. Because $|\vec{w}| \sin(\alpha)$ is the height of the parallelogram with base length $|\vec{v}|$, the area is $|\vec{v}||\vec{w}| \sin(\alpha)$ which is by the above formula equal to $|\vec{v} \times \vec{w}|$.
EXAMPLE. If $\vec{v} = (a, 0, 0)$ and $\vec{w} = (b \cos(\alpha), b \sin(\alpha), 0)$, then $\vec{v} \times \vec{w} = (0, 0, ab \sin(\alpha))$ which has length $|ab \sin(\alpha)|$.

ZERO CROSS PRODUCT. We see that $\vec{v} \times \vec{w}$ is zero if $\vec{v}$ and $\vec{w}$ are **parallel**.

ORIENTATION. The vectors $\vec{v}, \vec{w}$ and $\vec{v} \times \vec{w}$ form a **right handed coordinate system**. The right hand rule is: put the first vector $\vec{v}$ on the thumb, the second vector $\vec{w}$ on the pointing finger and the third vector $\vec{v} \times \vec{w}$ on the third middle finger.

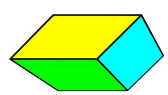


EXAMPLE. $\vec{i}, \vec{j}, \vec{i} \times \vec{j} = \vec{k}$ forms a right handed coordinate system.

DOT PRODUCT (is a scalar)		CROSS PRODUCT (is a vector)	
$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$	commutative	$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$	anti-commutative
$\lvert\vec{v} \cdot \vec{w}\rvert = \lvert\vec{v}\rvert\lvert\vec{w}\rvert \cos(\alpha)$	angle	$\lvert\vec{v} \times \vec{w}\rvert = \lvert\vec{v}\rvert\lvert\vec{w}\rvert \sin(\alpha)$	angle
$(a\vec{v}) \cdot \vec{w} = a(\vec{v} \cdot \vec{w})$	linearity	$(a\vec{v}) \times \vec{w} = a(\vec{v} \times \vec{w})$	linearity
$(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$	distributivity	$(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$	distributivity
$\{1,2,3\}.\{3,4,5\}$	in Mathematica	Cross[$\{1,2,3\}, \{3,4,5\}$]	in Mathematica
$\frac{d}{dt}(\vec{v} \cdot \vec{w}) = \dot{v} \cdot \vec{w} + \vec{v} \cdot \dot{w}$	product rule	$\frac{d}{dt}(\vec{v} \times \vec{w}) = \dot{\vec{v}} \times \vec{w} + \vec{v} \times \dot{\vec{w}}$	product rule

TRIPLE SCALAR PRODUCT. The scalar $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ is called the **triple scalar product** of $\vec{u}, \vec{v}, \vec{w}$.

PARALLELEPIPED. $[\vec{u}, \vec{v}, \vec{w}]$ is the volume of the parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$ because $h = \vec{u} \cdot \vec{n}/|\vec{n}|$ is the height of the parallelepiped if $\vec{n} = (\vec{v} \times \vec{w})$ is a normal vector to the ground parallelogram which has area $A = |\vec{n}| = |\vec{v} \times \vec{w}|$. The volume of the parallelepiped is $hA = \vec{u} \cdot \vec{n}|\vec{n}|/|\vec{n}| = |\vec{u} \cdot (\vec{v} \times \vec{w})|$.

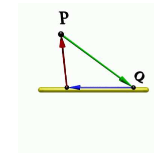


EXAMPLE. Find the volume of the parallel epiped which has the one corner $O = (1, 1, 0)$ and three corners $P = (2, 3, 1), Q = (4, 3, 1), R = (1, 4, 1)$ connected to it.
ANSWER: The parallelepiped is spanned by $\vec{u} = (1, 2, 1)$, $\vec{v} = (3, 2, 1)$, and $\vec{w} = (0, 3, 2)$. We get $\vec{v} \times \vec{w} = (1, -6, 9)$ and $\vec{u} \cdot (\vec{v} \times \vec{w}) = -2$. The volume is 2.

DISTANCE POINT-LINE (3D). If P is a point in space and $L = Q + t\vec{u}$ be the line which contains the vector $\vec{u}$ and the point Q, then

$$d(P, L) = |\vec{PQ} \times \vec{u}|/|\vec{u}|$$

is the distance between P and the line L. The representation of L as $r(t) = Q + t\vec{v}$ is called a **parametric equation** of the line.



PLANE THROUGH 3 POINTS P, Q, R:
The vector $\vec{n} = \vec{PQ} \times \vec{PR}$ is orthogonal to the plane. We will see next week that $\vec{n} = (a, b, c)$ defines the plane $ax + by + cz = d$, with $d = ax_0 + by_0 + cz_0$ which passes through the points $P = (x_0, y_0, z_0), Q, R$.

The cross product appears in many different applications:

ANGULAR MOMENTUM. If a mass point of mass m moves along a curve $\vec{r}(t)$, then the vector $\vec{L}(t) = m\vec{r}(t) \times \vec{r}\,'(t)$ is called the **angular momentum** of the point. It is coordinate system dependent.

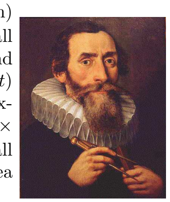
ANGULAR MOMENTUM CONSERVATION.

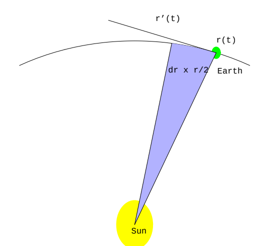
$$\frac{d}{dt}\vec{L}(t) = m\vec{r}\,'(t) \times \vec{r}\,'(t) + m\vec{r}(t) \times \vec{r}\,''(t) = \vec{r}(t) \times \vec{F}(t)$$

In a central field, where $\vec{F}(t)$ is parallel to $\vec{r}(t)$, we get $d/dtL(t) = 0$ which means $L(t)$ is constant.

TORQUE. The quantity $\vec{r}(t) \times \vec{F}(t)$ is also called the **torque**. The time derivative of the **momentum** $p = m\vec{r}\,'$ is the **force** F. the time derivative of the **angular momentum** $\vec{L} = \vec{r}(t) \times \vec{p}(t) = m\vec{r}(t) \times \vec{r}\,'(t)$ is the **torque**.

KEPLER'S AREA LAW. (Proof by Newton) The fact that $\vec{L}(t)$ is constant means first of all that $\vec{r}(t)$ stays in a plane spanned by $\vec{r}(0)$ and $\vec{r}\,'(0)$. The experimental fact that the vector $\vec{r}(t)$ sweeps over **equal areas in equal times** expresses angular momentum conservation: $|\vec{r}(t) \times \vec{r}\,'(t) dt/2| = |\vec{L} dt/m/2|$ is the area of a small triangle. The vector $\vec{r}(t)$ sweeps over an area $\int_0^T |\vec{L}| dt/(2m) = |\vec{L}|T/(2m)$ in time $[0, T]$.

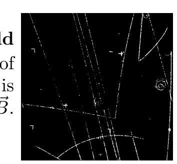




MORE PLACES IN PHYSICS WHERE THE CROSS PRODUCT OCCURS:

The **top**, the motion of a rigid body is describe by the angular momentum L and the angular velocity vector Ω in the body. Then $\dot{L} = L \times \Omega + M$, where M is an external **torque** obtained by external forces.

Electromagnetism: (informal) a particle moving along $\vec{r}(t)$ in a **magnetic field** $\vec{B}$ for example experiences the force $\vec{F}(t) = q\vec{r}\,'(t) \times \vec{B}$, where q is the charge of the particle. In a constant magnetic field, the particles move on circles: if m is the mass of the particle, then $m\vec{r}\,''(t) = q\vec{r}\,'(t) \times \vec{B}$ implies $m\vec{r}\,'(t) = q\vec{r}\,(t) \times \vec{B}$. Now $d/dt|\vec{r}|^2 = 2\vec{r} \cdot \vec{r}\,' = \vec{r} \cdot q\vec{r}\,(t) \times \vec{B} = 0$ so that $|\vec{r}|$ is constant.



Hurricanes are powerful storms with wind velocities of 74 miles per hour or more. On the northern hemisphere, hurricanes turn counterclockwise, on the southern hemisphere clockwise. This is a feature of all low pressure systems and can be explained by the Coriolis force. In a rotating coordinate system a particle of mass m moving along $\vec{r}(t)$ experience the following forces: $m\dot{\vec{\omega}}' \times \vec{r}$ (inertia of rotation), $2m\vec{\omega} \times \vec{r}\,'$ (Coriolis force) and $m\omega \times (\vec{\omega} \times \vec{r})$ (Centrifugal force). The Coriolis force is also responsible for the circulation in Jupiter's Red Spot.