

## CRITICAL POINTS

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**CRITICAL POINTS.** A point  $(x_0, y_0)$  in a region  $G$  is called a **critical point** of  $f(x, y)$  if  $\nabla f(x_0, y_0) = (0, 0)$ . Remarks. Critical points are also called **stationary points**. Critical points are candidates for extrema because at critical points, the directional derivative is zero. It is usually assumed that  $f$  is differentiable.

**EXAMPLE 1.**  $f(x, y) = x^4 + y^4 - 4xy + 2$ . The gradient is  $\nabla f(x, y) = (4(x^3 - y), 4(y^3 - x))$  with critical points  $(0, 0), (1, 1), (-1, -1)$ .

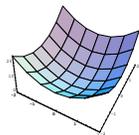
**EXAMPLE 2.**  $f(x, y) = \sin(x^2 + y) + y$ . The gradient is  $\nabla f(x, y) = (2x \cos(x^2 + y), \cos(x^2 + y) + 1)$ . For a critical points, we must have  $x = 0$  and  $\cos(y) + 1 = 0$  which means  $\pi + k2\pi$ . The critical points are at  $(0, \pi), (0, 3\pi), \dots$

**EXAMPLE 3.** ("volcano")  $f(x, y) = (x^2 + y^2)e^{-x^2 - y^2}$ . The gradient  $\nabla F = (2x - 2x(x^2 + y^2), 2y - 2y(x^2 + y^2))e^{-x^2 - y^2}$  vanishes at  $(0, 0)$  and on the circle  $x^2 + y^2 = 1$ . There are  $\infty$  many critical points.

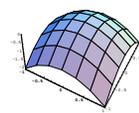
**EXAMPLE 4.** ("pendulum")  $f(x, y) = -g \cos(x) + y^2/2$  is the energy of the pendulum. The gradient  $\nabla F = (y, -g \sin(x))$  is  $(0, 0)$  for  $x = 0, \pi, 2\pi, \dots, y = 0$ . These points are equilibrium points, where the pendulum is at rest.

**EXAMPLE 5.** ("Volterra Lodka")  $f(x, y) = a \log(y) - by + c \log(x) - dx$ . (This function is left invariant by the flow of the Volterra Lodka differential equation  $\dot{x} = ax - bxy, \dot{y} = -cy + dxy$  which you might have seen in Math1b.) The point  $(c/d, a/b)$  is a critical point.

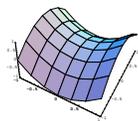
### TYPICAL EXAMPLES.



$$f(x, y) = x^2 + y^2$$

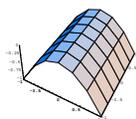


$$f(x, y) = -x^2 - y^2$$

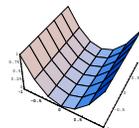


$$f(x, y) = x^2 - y^2$$

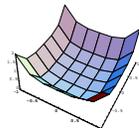
EXAMPLES WITH DISCRIMINANT  $D = \det(H) = 0$ .



$$f(x, y) = x^2$$



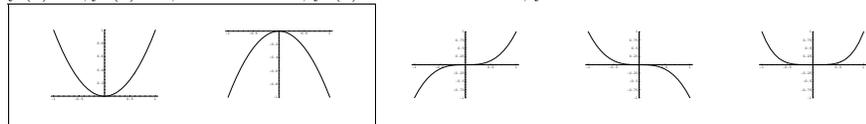
$$f(x, y) = -x^2$$



$$f(x, y) = x^4 + y^4$$

### CLASSIFICATION OF CRITICAL POINTS IN 1 DIMENSION.

$f'(x) = 0, f''(x) > 0$ , local minimum,  $f''(x) < 0$  local maximum,  $f'' = 0$  undetermined.



**CLASSIFICATION OF CRITICAL POINTS: SECOND DERIVATIVE TEST.** Let  $f(x, y)$  be a function of two variables with a critical point  $(x_0, y_0)$ . Define  $D = f_{xx}f_{yy} - f_{xy}^2$ , called the **discriminant** or **Hessian**.

(Remark: With the **Hessian matrix**  $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$  we can write  $D = \det(H)$  as a **determinant**.)

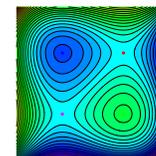
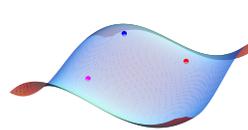
If  $D > 0$  and  $H_{11} > 0 \Rightarrow$  local minimum (bottom of valley)  
 If  $D > 0$  and  $H_{11} < 0 \Rightarrow$  local maximum (top of mountain).  
 If  $D < 0 \Rightarrow$  saddle point (mountain pass).

In the case  $D = 0$ , we would need higher derivatives to determine the nature of the critical point.

**EXAMPLE.** (A "napkin").

The function  $f(x, y) = x^3/3 - x - (y^3/3 - y)$  has the gradient  $\nabla f(x, y) = (x^2 - 1, -y^2 + 1)$ . It is the zero vector at the 4 critical points  $(1, 1), (-1, 1), (1, -1)$  and  $(-1, -1)$ . The Hessian matrix is  $H = f''(x, y) = \begin{bmatrix} 2x & 0 \\ 0 & -2y \end{bmatrix}$ .

$H(1, 1) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$	$H(-1, 1) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$	$H(1, -1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	$H(-1, -1) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$
$D = -4$ Saddle point	$D = 4, f_{xx} = -2$ Local maximum	$D = 4, f_{xx} = 2$ Local minimum	$D = -4$ Saddle point



**GLOBAL MAXIMA AND MINIMA.** To determine the maximum or minimum of  $f(x, y)$  on a domain, determine all critical points **in the interior the domain**, and compare their values with maxima or minima **at the boundary**.

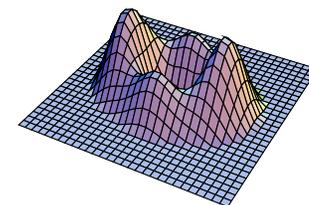
(A point is in the interior of  $G$ , if there is a small disc around  $(x_0, y_0)$  contained in  $G$ . A point is at the boundary of  $G$ , if any disc around  $(x_0, y_0)$  contains both points in  $G$  and in the complement).

**EXAMPLE 5.** Find the critical points of  $f(x, y) = 2x^2 - x^3 - y^2$ . With  $\nabla f(x, y) = 4x - 3x^2, -2y$ , the critical points are  $(4/3, 0)$  and  $(0, 0)$ . The Hessian is  $H(x, y) = \begin{bmatrix} 4 - 6x & 0 \\ 0 & -2 \end{bmatrix}$ . At  $(0, 0)$ , the discriminant is  $-8$  so that this is a saddle point. At  $(4/3, 0)$ , the discriminant is  $8$  and  $H_{11} = 4/3$ , so that  $(4/3, 0)$  is a local maximum.

### WHY DO WE CARE ABOUT CRITICAL POINTS?

- Critical points are candidates for extrema like maxima or minima.
- Knowing all the critical points and their nature tells alot about the function.
- Critical points are physically relevant. Examples are configurations with lowest energy).

**A CURIOUS OBSERVATION: (The island theorem)** Let  $f(x, y)$  be the height on an island. Assume there are only finitely many critical points on the island and all of them have nonzero determinant. Label each critical point with a  $+1$  "charge" if it is a maximum or minimum, and with  $-1$  "charge" if it is a saddle point. Sum up all the charges and you will get 1, independent of the function. This property is an example of an "index theorem", a prototype for important theorems in physics and mathematics.



**CRITICAL POINTS IN PHYSICS.** (informal) Most physical laws are based on the principle that the equations are critical points of a functional (in general in infinite dimensions).

- **Newton equations.** (Classical mechanics) A particle of mass  $m$  moving in a field  $V$  along a path  $\gamma : t \mapsto r(t)$  extremizes the integral  $S(\gamma) = \int_a^b m r'(t)^2/2 - V(r(t)) dt$ . Critical points  $\gamma$  satisfy the Newton equations  $m r''(t)/2 - \nabla V(r(t)) = 0$ .
- **Maxwell equations.** (Electromagnetism) The electromagnetic field  $(E, B)$  extremizes the Integral  $S(E, B) = \frac{1}{8\pi} \int (E^2 - B^2) dV$  over space time. Critical points are described by the the Maxwell equations in vacuum.
- **Einstein equations** (General relativity) If  $g$  is a dot product which depends on space and time, and  $R$  is the "curvature" of the corresponding curved space time, then  $S(g) = \int_R dV(g)$  is a function of  $g$  for which critical points  $g$  satisfy the Einstein equations in general relativity.

**OTHER WAYS TO FIND CRITICAL POINTS.** Some ideas: walk in the direction of the gradient until you reach a local maximum or walk backwards to reach a local minima. To find saddle points, consider the shortest path connecting two local minima and take the maximum along this path.