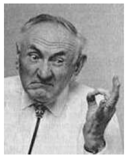
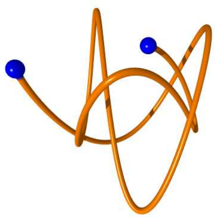
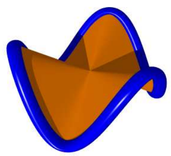
DIVERGENCE THEOREM II — Maths21a, O. Knill

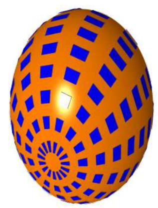
ZWICKY'S VIEW. Fritz Zwicky was an astronomer who also devised the **morphological method** for creativity. The method is one of many tools to solve problems which require new ideas. Zwicky was highly creative himself. The idea of the supernovae, dark matter and many pulsar discoveries can be attributed to him. In 1948 for example, Zwicky suggested to use extraterrestrial sources to reconstruct the universe. This should begin with changing other planets, moons and asteroids by making them inhabitable and to change their orbits around the sun in order to adjust their temperature. He even suggested to alter the fusion of the sun to make displace our own space solar system towards an other planetary system. Zwickies **morphological method** is to organize ideas in boxes like a spreadsheet. For example, if we wanted to make order in the zoo of integral theorems we have seen now, we would one coordinate to display the dimension, in which we work and the second coordinate the maximal dimension along which we integrate in the theorem.



	1	2	3
1	Fundamental theorem of calculus	-	-
2	Fundamental theorem of line integrals	Greens theorem	-
3	Fundamental theorem of line integrals	Stokes theorem	Gauss theorem







Fundamental thm of line integrals. We integrate over 1 and 0 dimensional objects. The 0 dimensional object has no boundary.

Green and Stokes theorems. We integrate over 2 and 1 dimensional objects. The 1 dimensional object is a closed curve and has no boundary.

Divergence theorem. We integrate over 3 and 2 dimensional objects. The 2 dimensional object is a closed surface and has no boundary.

One could build now for each of the differential operators grad, curl and div such a matrix and see whether there is a theorem. Many combinations do not make sense like integrating the curl over a three dimensional object. But there is a theorem, which seem have escaped: if we integrate in two dimensions the $\text{div}(F) = M_x + N_y$ over a region R, then there this can be written as an integral over the boundary. There is indeed such a theorem, but it is just a version of Greens theorem in disguise (turn the vector field by 90 degrees and replace the line by a 2D version of the flux integral). It can be seen as a special case of the divergence theorem in three dimensions and it does not make sense to put it on the footing of the other theorems. Anyway, the morphological method was used in management like in Ciba, it is today just one of a variety of methods to come up with new ideas.

IDENTITIES. While direct computations can verify the identities to the left, they become evident with **Nabla calculus** from formulas for vectors like $\vec{v} \times \vec{v} = \vec{0}, \vec{v} \cdot \vec{v} \times \vec{w} = 0$ or $u \times (v \times w) = v(u \cdot w) - (u \cdot v)w$.

$\text{div}(\text{curl}(F)) = 0$.
$\text{curl}(\text{grad}(F)) = \vec{0}$
$\text{curl}(\text{curl}(F)) = \text{grad}(\text{div}(F) - \Delta(F))$.

$\nabla \cdot \nabla \times F = 0$.
$\nabla \times \nabla F = \vec{0}$.
$\nabla \times \nabla \times F = \nabla(\nabla \cdot F) - (\nabla \cdot \nabla)F$.

QUIZZ. Is there a vector field G such that $F = (x + y, z, y^2) = \text{curl}(G)$? Answer: no because $\text{div}(F) = 1$ is incompatible with $\text{divcurl}(G)) = 0$.

BOXED OVERVIEW. All integral theorems are of the form $\int_R F' = \int_{\delta R} F$, where F' is a "derivative" and δR is a "boundary". There are 2 such theorems in dimensions 2, three theorems in dimensions 3, four in dimension 4 etc.

dim	dim(R)	theorem
1D	1	Fund. thm of calculus

2D	1	Fund. thm of line integrals
2D	2	Green's theorem

dim	dim(R)	theorem
3D	1	Fundam. thm of line integrals
3D	2	Stokes theorem
3D	3	Divergence theorem

$1 \mapsto 1$	f'	derivative
$1 \mapsto 2$	∇f	gradient
$2 \mapsto 1$	$\nabla \times F$	curl

$1 \mapsto 3$	∇f	gradient
$3 \mapsto 3$	$\nabla \times F$	curl
$3 \mapsto 1$	$\nabla \cdot F$	divergence

MAXWELL EQUATIONS. c is the speed of light.

$\text{div}(B) = 0$	No monopoles	there are no magnetic monopoles.
$\text{curl}(E) = -\frac{1}{c} B_t$	Faraday's law	change of magnetic flux induces voltage
$\text{curl}(B) = \frac{1}{c} E_t + \frac{4\pi}{c} j$	Ampère's law	current or change of E produces magnetic field
$\text{div}(E) = 4\pi\rho$	Gauss law	electric charges are sources for electric field

MAGNETOSTATICS: $\text{curl}(B) = 0$ so that $B = \text{grad}(f)$. But since also $\text{div}(B) = 0$, we have

$$\Delta f = \text{div}(\text{grad}(f)) = 0$$

ELECTROSTATICS. $\text{curl}(E) = 0$ so that $E = \text{grad}(f)$. But since also $\text{div}(E) = 0$, we have

$$\Delta f = \text{div}(\text{grad}(f)) = 0$$

Static electric and magnetic fields have a harmonic potential.

FLUID DYNAMICS. v velocity, ρ density of fluid.

Continuity equation	$\dot{\rho} + \text{div}(\rho v) = 0$	no fluid get lost
Incompressibility	$\text{div}(v) = 0$	incompressible fluids
Irrotational	$\text{curl}(v) = 0$	no vorticity
Potential fluids	$v = \text{grad}(f), \Delta(f) = 0$	incompressible, irrotational fluids

Incompressible, irrotational fluid velocity fields have a potential which is harmonic

HARMONIC FUNCTIONS. A function $f(x, y, z)$ is called **harmonic**, if $\Delta f(x, y, z) = 0$. Harmonic functions play an important role in physics. For example time independent solutions of the Schrödinger equation $i\hbar f_t = \Delta f$ or the heat equation $f_t = \Delta f$ are harmonic. In fluid dynamics, incompressible and irrotational fluids with velocity distribution v satisfy $v = \text{grad}(f)$ and since $\text{div}(v) = 0$, also $\Delta f = 0$. Harmonic functions have properties which can be seen nicely using the divergence theorem.

MAXIMUM PRINCIPLE. A harmonic function in D can not have a local maximum inside D.

MEAN VALUE PROPERTY. The average of a harmonic function on a sphere of radius r around a point P is equal to the value of the function at the point P.

These properties are direct consequences of the divergence theorem: for example, if f would have a maximum, then the gradient field $F = \text{grad}(f)$ near P points towards P because the gradient points into the direction of maximal increase. If we place a small sphere around P then the vector field F enters the sphere at every point. The flux of F through the sphere is negative. By the divergence theorem, the divergence of F integrated over the interior of the sphere must be negative. But the divergence of F is the Laplacian of f which is zero.

THE PIGEON PROBLEM. A farmer drives a closed van through the countryside. Inside the van are dozens of pigeons. The van weights 200 pounds, the pigeon, an other 100 pounds. The driver has to pass a bridge, which can only sustain 250 pounds. The driver has an idea: if all the pigeon would fly while crossing the bridge, a passage would be possible. Question: does the bridge hold? What happens if the cage is open?

The flight of the pigeon produces a wind distribution in the cage. The question is whether the flying pigeon will still produces a weight on the van.