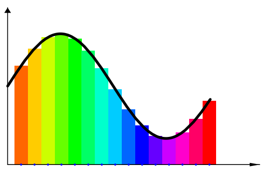
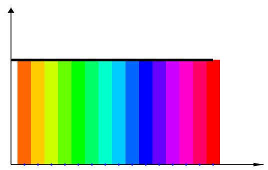
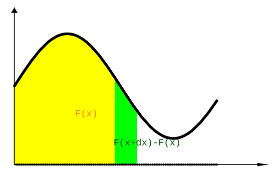
2D INTEGRALS	**Math21a, O. Knill**

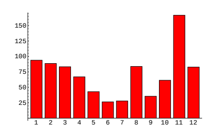
1D INTEGRATION IN 100 WORDS. If $f(x)$ is a continuous function then $\int_a^b f(x)\ dx$ can be defined as a limit of the **Riemann sum** $f_n(x) = \frac{1}{n}\sum_{x_k \in [a,b]} f(x_k)$ for $n \to \infty$ with $x_k = k/n$. This integral divided by $|b-a|$ is the **average** of f on $[a,b]$. The integral $\int_a^b f(x)\ dx$ can be interpreted as an **signed area** under the graph of f, which can be negative too. If $f(x) = 1$, the integral is the **length** of the interval. The function $F(x) = \int_a^x f(y)\ dy$ is called an **anti-derivative** of f. The **fundamental theorem of calculus** states $F'(x) = f(x)$. Unlike the derivative, anti-derivatives can not always be expressed in terms of known functions. An example is: $F(x) = \int_0^x e^{-t^2}\ dt$. Often, the anti-derivative can be found: Example: $f(x) = \cos^2(x) = (\cos(2x)+1)/2, F(x) = x/2 - \sin(2x)/4$.

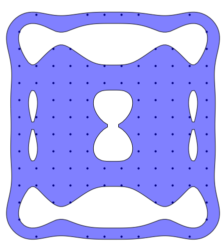
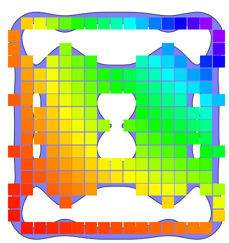
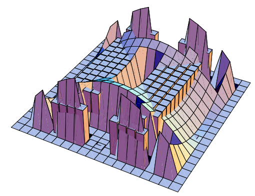
AVERAGES=MEAN. www.worldclimate.com gives the following data for the average monthly rainfall (in mm) for Cambridge, MA, USA (42.38 North 71.11 West,18m Height).

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
93.9	88.6	83.3	67.0	42.9	26.4	27.9	83.8	35.5	61.4	166.8	82.8

The average $860.3/12 = 71.7$ is a Rieman sum integral.



2D INTEGRATION. If $f(x,y)$ is a continuous function of two variables on a region R, the integral $\int_R f(x,y)\ dxdy$ can be defined as the limit $\frac{1}{n^2}\sum_{i,j,x_{i,j}\in R} f(x_i,y_j)$ with $x_{i,j} = (i/n, j/n)$ when n goes to infinity. If $f(x,y) = 1$, then the integral is the **area** of the region R. The integral divided by the area of R is the **average** value of f on R. For many regions, the integral can be calculated as a **double integral** $\int_a^b [\int_{c(x)}^{d(x)} f(x,y)\ dy]dx$. In general, the region must be split into pieces, then integrated seperately.

One can interpret $\int\int_R f(x,y)\ dydx$ as the volume of solid below the graph of f and above R in the $x-y$ plane. (As in 1D integration, the volume of the solid below the x-y plane is counted negatively).

EXAMPLE. Calculate $\int\int_R f(x,y)\ dxdy$, where $f(x,y) = 4x^2y^3$ and where R is the rectangle $[0,1]\times[0,2]$.

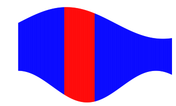
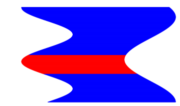
$$\int_0^1 [\int_0^2 4x^2y^3\ dy]\ dx = \int_0^1 [x^2y^4|_0^2]\ dx = \int_0^1 x^2(16-0)\ dx = 16x^3/3|_0^1 = \frac{16}{3}\ .$$

FUBINI'S THEOREM. $\int_a^b\int_c^d f(x,y)\ dxdy = \int_c^d\int_a^b f(y,x)\ dydx.$

TYPES OF REGIONS.

$\int\int_R f\ \ dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ \ dydx$
type I region.

$\int\int_R f\ \ dA = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x,y)\ \ dxdy$
type II region.

EXAMPLE. Let R be the triangle $1 \geq x \geq 0, 1 \geq y \geq 0, y \leq x$. Calculate $\int\int_R e^{-x^2}\ dxdy$.

ATTEMPT. $\int_0^1 [\int_y^1 e^{-x^2}\ dx]dy$. We can not solve the inner integral because e^{-x^2} has no anti-derivative in terms of elementary functions.

IDEA. Switch order: $\int_0^1 [\int_0^x e^{-x^2}\ dy]\ dx = \int_0^1 xe^{-x^2}\ dx = -\frac{e^{-x^2}}{2}|_0^1 = \frac{(1-e^{-1})}{2} = 0.316....$

A special case of switching the order of integration is **Fubini's theorem.**

If you can't solve a double integral, try to change the order of integration!

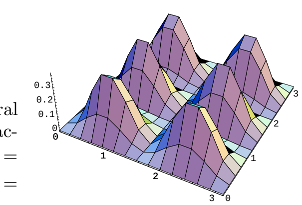
QUANTUM MECHANICS. In quantum mechanics, the motion of a particle (like an electron) in the plane is determined by a function $u(x,y)$, the wave function. Unlike in classical mechanics, the position of a particle is given in a probabilistic way only. If R is a region and u is normalized so that $\int |u|^2\ dxdy = 1$, then $\int_R |u(x,y)|^2\ dxdy$ is the **probability**, that the particle is in R.

EXAMPLE. Unlike a classical particle, a quantum particle in a box $[0,\pi]\times[0,\pi]$ can have a discrete set of energies only. This is the reason for the name "quantum". If $-(u_{xx}+u_{yy}) = \lambda u$, then a particle of mass m has the energy $E = \lambda\hbar^2/2m$. A function $u(x,y) = \sin(kx)\sin(ny)$ represents a particle of energy $(k^2+n^2)\hbar^2/(2m)$. Let us assume $k=2$ and $n=3$ from now on. Our aim is to find the probability that the particle with energy $13\hbar^2/(2m)$ is in the middle 9'th $R = [\pi/3, 2\pi/3]\times[\pi/3, 2\pi/3]$ of the box.

SOLUTION: We first have to normalize $u^2(x,y) = \sin^2(2x)\sin^2(3y)$, so that the average over the whole square is 1:

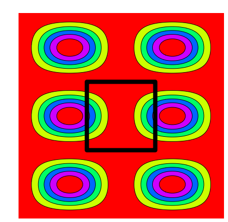
$$A = \int_0^\pi \int_0^\pi \sin^2(2x)\sin^2(3y)\ dxdy\ .$$

To calculate this integral, we first determine the inner integral $\int_0^\pi \sin^2(2x)\sin^2(3y)\ dx = \sin^2(3y)\int_0^\pi \sin^2(2x)\ dx = \frac{\pi}{2}\sin^2(3y)$ (the factor $\sin^2(3y)$ is treated as a constant). Now, $A = \int_0^\pi (\pi/2)\sin^2(3y)\ dy = \frac{\pi^2}{4}$, so that the **probability amplitude function** is $f(x,y) = \frac{4}{\pi^2}\sin^2(2x)\sin^2(3y)$.



The probability that the particle is in R is slightly smaller than 1/9:

$$\begin{aligned} \frac{1}{A}\int_R f(x,y)\ dxdy &= \frac{4}{\pi^2}\int_{\pi/3}^{2\pi/3}\int_{\pi/3}^{2\pi/3} \sin^2(2x)\sin^2(3y)\ dxdy \\ &= \frac{4}{\pi^2}(4x - \sin(4x))/8|_{\pi/3}^{2\pi/3}(6x - \sin(6x))/12|_{\pi/3}^{2\pi/3} \\ &= 1/9 - 1/(4\sqrt{3}\pi) \end{aligned}$$



The probability is slightly smaller than 1/9.

WHERE DO DOUBLE INTEGRALS OCCUR?
- compute areas.
- compute averages. Examples: average rain fall or average population in some area.
- probabilities. Expectation of random variables.
- quantum mechanics: probability of particle being in a region. - find moment of inertia $\int\int_R (x^2+y^2)\rho(x,y)dxdy$
- find center of mass $(\int\int_R x\rho(x,y)\ dxdy/M, \int\int_R y\rho(x,y)\ dxdy/M)$, with $M = \int\int_R dxdy$.
- compute some 1D integrals.

TRIPLE INTEGRALS are defined similarly and covered in detail later. The area under a graph of $f(x,y)$ can be written as a triple integral. Fubinis theorem generalizes and $\int\int\int_R 1\ dxdydz$ is a **volume**.